Boundary Value Problems on Weighted Networks

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Abstract

We present here a systematic study of general boundary value problems on weighted networks that includes the variational formulation of such problems. In particular, we obtain the discrete version of the Dirichlet Principle and we apply it to the analysis of the inverse problem of identifying the conductivities of the network in a very general framework. Our approach is based on the development of an efficient vector calculus on weighted networks which mimetizes the calculus in the smooth case. The key tool is an adequate construction of the tangent space at each vertex. This allows us to consider discrete vector fields, inner products and general metrics. Then, we obtain discrete versions of derivative, gradient, divergence and Laplace-Beltrami operators, satisfying analogous properties to those verified by their continuous counterparts. On the other hand we develop the corresponding integral calculus that includes the discrete versions of the Integration by Parts technique and Green's Identities. Finally, we apply our discrete vector calculus to analyze the consistency of difference schemes used to solve numerically a Robin boundary value problem in a square.

Key Words: Weighted networks, Vector Calculus, Discrete operators, Discrete Green's Identities, Discrete boundary value problems, Inverse problem, Difference schemes.

1 Introduction

The discrete vector calculus theory is a very fruitful area of work in many mathematical branches not only for its intrinsic interest but also for its applications, [2, 5, 7, 13, 18, 19]. One can construct a discrete vector calculus by considering simplicial complexes that approximates locally a smooth manifold and then the Whitney application can be used to define inner products on the cochain spaces. This gives rise to a combinatorial Hodge theory that allows to translate the basic notions of Riemannian geometry into combinatorial terms and that shows that the combinatorial objects are good approximations for the smooth ones, [11].

Alternatively, one can approximate a smooth manifold by means of non-simplicial meshes and then one can define discrete operators either by truncating the smooth ones or interpolating on the mesh elements. This approach is considered in the aim of mimetic methods which are used in the context of difference schemes to solve numerically boundary values problems. These methods have good computational properties, [12, 13]. Another approach is to deal with the mesh as the unique existent space and then the discrete vector calculus is described throughout tools from the

Algebraic Topology since the geometric realization of the mesh is a unidimensional CW-complex. So, the discrete operators can be defined in combinatorial terms, [9, 19].

Our work falls within the last ambit but, instead of importing the tools from Algebraic Topology, we construct the discrete vector calculus from the graph structure itself following the guidelines of Differential Geometry. The key to develop our discrete calculus is an adequate construction of the tangent space at each vertex of the graph. The concepts of discrete vector fields and bilinear forms are a likely result of the definition of tangent space. Moreover, they are general, while only orthogonal bilinear forms and vector fields that are either symmetric or antisymmetric are habitually considered in the literature, [5, 7]. We obtain discrete versions of the derivative, gradient, divergence and Laplace-Beltrami operators that satisfy the same properties that its continuum analogues.

We also develop an integral calculus that includes the discrete versions of Integration by Parts formulae, Divergence Theorem and the Green's Identities. As a consequence we describe appropriately general boundary value problems on arbitrary nonempty subsets of weighted networks as well as its variational formulation. Then, we obtain necessary and sufficient conditions for the existence and uniqueness of solution. Moreover, we give the discrete version of the Dirichlet Principle for self-adjoint boundary value problems associated with elliptic operators. As an application we obtain a generalization of the inverse problem of identifying the conductivity between nodes in the network that has been considered in [7]. Finally, we apply our discrete vector calculus to analyze the consistence of difference schemes used to solve numerically a Robin boundary value problem in a square. We show that any difference scheme is completely determined by a vector field and a field of endomorphisms and we also show that these fields induce, in a natural way, a discretization of the co-normal derivative. Therefore, special properties of the difference schemes such as consistency and positivity can be characterized in terms of the fields.

2 Preliminaries

Throughout the paper, $\Gamma = (V, E)$ denotes a simple connected and locally finite graph without loops, with vertex set V and edge set E. Two different vertices, $x, y \in V$, are called *adjacent*, which is represented by $x \sim y$, if $\{x, y\} \in E$. In this case, the edge $\{x, y\}$ is also denoted as e_{xy} and the vertices x and y are called incidents with e_{xy}. In addition, for any $x \in V$ the value $k(x)$ denote the number of vertices adjacent to x. Moreover, $d(x, y)$ is the length of the shortest path joining x and y and it is well-known that d defines a distance on the graph.

Given a vertex subset $F \subset V$, we denote by F^c its complement in V and by χ_F its characteristic function. Moreover, the sets $\overset{\circ}{F} = \{x \in F : \{y : d(x, y) = 1\} \subset F\}$, $\delta(F) = \{x \in V : d(x, F) = 1\}$ and $\bar{F} = F \cup \delta(F)$, are called *interior, boundary and closure of* F, respectively.

We denote by $\mathcal{C}(V)$, $\mathcal{C}(V \times V)$ and $\mathcal{C}(V \times V \times V)$, the vector spaces of real functions defined on the sets that appear between brackets. If $u \in \mathcal{C}(V)$, $f \in \mathcal{C}(V \times V)$ and $a \in \mathcal{C}(V \times V \times V)$, uf and $u\,a$ denote the functions defined for any $x, y, z \in V$ as $(uf)(x, y) = u(x)f(x, y)$ and $(u\,a)(x, y, z) = u(x)a(x, y, z)$, respectively. In addition, given $F \subset V$ and $f \in \mathcal{C}(V \times V)$ we call *the restriction of* f *on* F, the function $f_F \in \mathcal{C}(V \times V)$ given by $f_F(x, y) = f(x, y)$, when $(x, y) \in \bar{F} \times \bar{F} \setminus \delta(F) \times \delta(F)$ and $f_F(x, y) = 0$, otherwise.

If $u \in \mathcal{C}(V)$, the *support of* u is the set $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. The vector space formed by the functions in $\mathcal{C}(V)$ with finite support is denoted by $\mathcal{C}_0(V)$. Clearly $\mathcal{C}(V) = \mathcal{C}_0(V)$ iff Γ is a finite graph; *i.e.*, iff V is a finite set. In addition, if $F \subset V$, $\mathcal{C}(F)$ denotes the subspace of $\mathcal{C}(V)$ formed by the functions whose support is contained in F.

A function $\nu \in \mathcal{C}(V)$ is called a *weight on* V if $\nu(x) > 0$ for all $x \in V$. For each weight ν on V and any $u \in \mathcal{C}_0(V)$ we denote by $\displaystyle\int_V u\, d\nu$ the value $\displaystyle\sum_{x\in V} u(x)\,\nu(x)$. In particular, when $\nu(x) = 1$ for any $x \in V$, $\displaystyle\int_V u\, d\nu$ is simply denoted by $\displaystyle\int_V u\, dx$.

Throughout the paper we make use of the following subspaces of $\mathcal{C}(V \times V)$ and $\mathcal{C}(V \times V \times V)$:

$$\mathcal{C}(\Gamma) = \{f \in \mathcal{C}(V \times V) : f(x, y) = 0, \quad \text{if } d(x, y) \neq 1\},$$

$$\mathcal{C}(\Gamma \times \Gamma) = \{f \in \mathcal{C}(V \times V \times V) : f(x, y, z) = 0, \quad \text{if } d(x, y) \cdot d(x, z) \neq 1\}.$$

Next we define the tangent space at a vertex of a graph, see [2] for its definition in the case of grids. Given $x \in V$, we call the real vector space of formal linear combinations of the edges incident with x, *tangent space at* x and we denote it by $T_x(\Gamma)$. So, the set of edges incident with x is a basis of $T_x(\Gamma)$, that is called *coordinate basis of* $T_x(\Gamma)$ and hence, $\dim T_x(\Gamma) = k(x)$. Note that, in the discrete setting, the dimension of the tangent space varies with each vertex except when the graph is regular.

We call any application $\mathsf{f} : V \longrightarrow \displaystyle\bigcup_{x\in V} T_x(\Gamma)$ such that $\mathsf{f}(x) \in T_x(\Gamma)$ for each $x \in V$, *vector field on* Γ. The *support of* f is defined as the set $\mathsf{supp}(\mathsf{f}) = \{x \in V : \mathsf{f}(x) \neq \mathsf{0}\}$. The spaces of vector fields and vector fields with finite support on Γ are denoted by $\mathcal{X}(\Gamma)$ and $\mathcal{X}_0(\Gamma)$, respectively.

If f is a vector field on Γ, then f is uniquely determined by its components in the coordinate basis. Therefore, we can associate with f the function $f \in \mathcal{C}(\Gamma)$ such that for each $x \in V$, $\mathsf{f}(x) = \displaystyle\sum_{y\sim x} f(x, y)\, e_{xy}$ and hence $\mathcal{X}(\Gamma)$ can be identified with $\mathcal{C}(\Gamma)$.

A vector field f is called a *flow* when its component function satisfies that $f(x, y) = -f(y, x)$ for any $x, y \in V$. If $\mathsf{f} \in \mathcal{X}(\Gamma)$ we call *flow determined by* f the field $\hat{\mathsf{f}}$ whose component function is $\hat{f}(x, y) = \frac{1}{2}\,(f(x, y) - f(y, x))$, where f is the component function of f. Clearly $\hat{\mathsf{f}} = \mathsf{f}$ iff f is a flow and $\hat{\mathsf{f}} \in \mathcal{X}_0(\Gamma)$ when $\mathsf{f} \in \mathcal{X}_0(\Gamma)$. More generally, if $\nu \in \mathcal{C}(V)$ is a weight on V, a vector f is called a *ν-flow* when the vector field $\nu\mathsf{f}$ is a flow.

If $u \in \mathcal{C}(V)$ and $\mathsf{f} \in \mathcal{X}(\Gamma)$ has $f \in \mathcal{C}(\Gamma)$ as its component function, the field $u\mathsf{f}$ is defined as the field whose component function is uf. In addition, when $F \subset V$ the *restriction of* f *on* F is the vector field f_F whose component function is f_F. Moreover, $\mathsf{supp}(\mathsf{f}_F) \subset \bar{F}$ and f_F is a ν-flow when f is.

If $\mathsf{f}, \mathsf{g} \in \mathcal{X}(\Gamma)$ and $f, g \in \mathcal{C}(\Gamma)$ are their component functions, the expression $\langle \mathsf{f}, \mathsf{g}\rangle$ denotes the function in $\mathcal{C}(V)$ given by

$$\langle \mathsf{f}, \mathsf{g}\rangle(x) = \int_V f(x, y)g(x, y)\, dy, \quad \text{for any } x \in V. \tag{1}$$

Clearly, for any $x \in V$, $\langle\cdot,\cdot\rangle(x)$ determines an inner product on $T_x(\Gamma)$. So, if for each $x \in V$, $\mathcal{T}_x^2(\Gamma)$ is the vector space of bilinear forms on $T_x(\Gamma)$, the application $\langle\cdot,\cdot\rangle\colon V \longrightarrow \displaystyle\bigcup_{x\in V} \mathcal{T}_x^2(\Gamma)$ can be considered as a metric on Γ that we call *the canonical metric*.

If for each $x \in V$, we consider $\mathcal{T}_x^1(\Gamma)$ the vector space of endomorphisms on $T_x(\Gamma)$, we call any application $\mathsf{A}\colon V \longrightarrow \bigcup\limits_{x\in V} \mathcal{T}_x^1(\Gamma)$ such that for any $x \in V$, $\mathsf{A}(x) \in \mathcal{T}_x^1(\Gamma)$, *field of endomorphisms on* Γ. The vector space of fields of endomorphisms on Γ is denoted by $\mathcal{T}^1(\Gamma)$.

If $\mathsf{A} \in \mathcal{T}^1(\Gamma)$, its *component function* is the function $a \in \mathcal{C}(\Gamma \times \Gamma)$ such that

$$a(x, y, z) = \langle \mathsf{A}(x)e_{xz}, e_{xy}\rangle(x), \quad \text{for any } x \in V \text{ and } y \sim x,\ z \sim x. \tag{2}$$

If $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ and $\mathsf{f} \in \mathcal{X}(\Gamma)$ we define Af the vector field whose component function is given by $h(x, y) = \sum\limits_{z\in V} a(x, y, z)f(x, z)$, for any $x, y \in V$, where $a \in \mathcal{C}(\Gamma\times\Gamma)$ and $f \in \mathcal{C}(\Gamma)$ are the component functions of A and f, respectively.

If $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ and $a \in \mathcal{C}(\Gamma \times \Gamma)$ is its component function, we say that the field A is *diagonal, symmetric or positive (semi)-definite* if for any $x \in V$, the matrix $\Big(a(x, y, z)\Big)_{\substack{y\sim x\\ z\sim x}}$ has the same property and we say that A is invertible iff $\mathsf{A}(x)$ is for any $x \in V$. In this case, we denote by A^{-1} the field of endomorphisms defined as $\mathsf{A}^{-1}(x) = (\mathsf{A}(x))^{-1}$. Moreover, when A is symmetric and positive definite the expression $\langle \mathsf{Af}, \mathsf{g}\rangle$ determines a metric on Γ. Observe that A is a diagonal field of endomorphisms iff $a(x, y, z) = 0$ for any x, y, z such that $y \neq z$.

The triple (Γ, μ, ν), where μ, ν are two weights on V, is called *weighted graph*. We always suppose that a weighted graph is endowed with the canonical metric. So, on a weighted graph we can consider the following inner products on $\mathcal{C}_0(V)$ and on $\mathcal{X}_0(\Gamma)$,

$$\int_V u\, v\, d\nu, \quad u, v \in \mathcal{C}_0(V) \quad \text{and} \quad \frac{1}{2}\int_V \langle \mathsf{f}, \mathsf{g}\rangle\, d\mu, \quad \mathsf{f}, \mathsf{g} \in \mathcal{X}_0(\Gamma), \tag{3}$$

where the factor $\frac{1}{2}$ is due to the fact that each edge is considered twice. In addition, the above expressions are also valid when only one of the functions or one of the vector fields has finite support.

3 Difference operators on weighted graphs

Our objective in this section is to define the discrete analogues of the fundamental first and second order differential operators on Riemannian manifolds, specifically the derivative, gradient, divergence and the composition of the divergence with a linear operator that acts on the derivative. The last one is called *second order difference operator* whereas the former are generically called *first order difference operators.*

From now on we suppose fixed the weighted graph (Γ, μ, ν) and also the associated inner products on $\mathcal{C}_0(V)$ and $\mathcal{X}_0(\Gamma)$.

We call *derivative operator* the linear application $\mathsf{d}\colon \mathcal{C}(V) \longrightarrow \mathcal{X}(\Gamma)$ that assigns to any $u \in \mathcal{C}(V)$ the flow $\mathsf{d}u$, called *derivative of* u, given by

$$(\mathsf{d}u)(x) = \sum_{y\sim x} \Big(u(y) - u(x)\Big)\, e_{xy}. \tag{4}$$

Clearly, it is verified that $\mathsf{d}u = 0$ iff u is a constant function.

We define the *divergence operator* as the linear application $\mathsf{div}: \mathcal{X}(\Gamma) \longrightarrow \mathcal{C}(V)$ that assigns to any $\mathsf{f} \in \mathcal{X}(\Gamma)$ the function $\mathsf{div}\,\mathsf{f}$, called *divergence of* f, determined by the relation

$$\int_V u\, \mathsf{div}\,\mathsf{f}\, d\nu = -\frac{1}{2}\int_V \langle \mathsf{d}u, \mathsf{f}\rangle\, d\mu, \quad \text{for any } u \in \mathcal{C}_0(V). \tag{5}$$

Therefore, if $f \in \mathcal{C}(\Gamma)$ denotes the component function of f, then

$$\mathsf{div}\,\mathsf{f}(x) = \frac{1}{\nu(x)}\int_V \widehat{(\mu f)}(x,y)\, dy, \tag{6}$$

which in particular implies that $\mathsf{div}\,\mathsf{f} \in \mathcal{C}_0(V)$ when $\mathsf{f} \in \mathcal{X}_0(\Gamma)$. Observe that Identity (5) says that $\mathsf{div} = -\mathsf{d}^*$ with respect to the inner products given on $\mathcal{C}_0(V)$ and $\mathcal{X}_0(\Gamma)$. In addition, when $\mathsf{f} \in \mathcal{X}_0(\Gamma)$ Identity (5) is also valid for any $u \in \mathcal{C}(V)$. When $\nu = \mu = 1$, then the corresponding divergence operator will be denoted by Div. Clearly for any weights ν, μ it is verified that $\mathsf{div}\,\mathsf{f} = \frac{1}{\nu}\mathsf{Div}\,(\mu\mathsf{f})$, for any $\mathsf{f} \in \mathcal{X}(\Gamma)$.

Now we introduce the fundamental second order difference operators on $\mathcal{C}(V)$ which are obtained by composition of two first order operators. For each field of endomorphisms A consider the endomorphism of $\mathcal{C}(V)$ given by $\mathcal{L}(u) = -\mathsf{div}\,(\mathsf{A}\mathsf{d}u)$, for any $u \in \mathcal{C}(V)$. When A is symmetric and positive definite, then Ad can be interpreted as the *gradient operator* associated with the metric determined by A^{-1} and hence $\mathcal{L}$ is nothing else than the *Laplace-Beltrami operator* associated with this metric.

Given $\mathsf{A} \in \mathcal{T}^1(\Gamma)$, A^* denotes the field of endomorphism that assigns to any $x \in V$ the transpose of $\mathsf{A}(x)$ and then we define the endomorphism $\mathcal{L}^*(u) = -\mathsf{div}\,(\mathsf{A}^*\mathsf{d}u)$. The definition of the above second order difference operator leads directly to the identities

$$\int_V v\,\mathcal{L}(u)\, d\nu = \frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}u, \mathsf{d}v\rangle\, d\mu = \int_V u\,\mathcal{L}^*(v)\, d\nu, \quad \text{for any } u, v \in \mathcal{C}_0(V) \tag{7}$$

that are also valid when only one of the functions is in $\mathcal{C}_0(V)$. In particular, $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ on $\mathcal{C}_0(V)$ and $\mathcal{L}$ is self-adjoint when A is a symmetric field of endomorphisms. Observe that taking $v = 1$ in Identity (7) we obtain that

$$\int_V \mathcal{L}(u)\, d\nu = 0, \quad \text{for any } u \in \mathcal{C}_0(V). \tag{8}$$

Identity (7) leads us to generalize the concept of elliptic operator introduced by Y. Colin de Verdière in [8]. So, we say that the operator $\mathcal{L}$ is *semi-elliptic* when it is self-adjoint and positive semi-definite on $\mathcal{C}_0(V)$ and *elliptic* when, in addition, $u \in \mathcal{C}_0(V)$ verifies that $\mathcal{L}(u) = 0$ iff u is constant and hence $u = 0$ when Γ is not finite. The above notions do not depend on the weight ν and for this reason the pair (A, μ) is called semi-elliptic or elliptic when the operator $\mathcal{L}$ is semi-elliptic or elliptic, respectively. Clearly, if A is a symmetric and positive semi-definite field of endomorphisms, then the pair (A, μ) is semi-elliptic for any weight μ and it is elliptic if, in addition, A is a positive definite field.

Our next aim is to obtain an explicit expression of $\mathcal{L}(u)$ for any $u \in \mathcal{C}(V)$. From Identity (7), and keeping in mind that $\mathcal{L}$ is a linear operator, it is clear that

$$\mathcal{L}(u)(x) = \frac{1}{2\nu(x)}\sum_{y \in V} u(y) \int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle\, d\mu, \quad \text{for any } x \in V.$$

Therefore, we define the *coefficient function of the pair* (A,μ) as the function $c\colon V\times V\longrightarrow\mathbb{R}$ given by $c(x,x)=0$ for any $x\in V$ and

$$c(x,y)=-\frac{1}{2}\int_V\langle\mathsf{A}\mathsf{d}\varepsilon_y,\mathsf{d}\varepsilon_x\rangle\,d\mu,\qquad\text{if}\quad x\neq y. \tag{9}$$

As we will prove in the next lemma $c(x,y)=0$ when $d(x,y)>2$, which reflects a locality property of the operator $\mathcal{L}$. Observe that if we denote by c^* the coefficient function of the pair (A^*,μ), then Identity (9) implies that $c(y,x)=c^*(x,y)$ for any $x,y\in V$ and in particular, that $c=c^*$ when A is a symmetric field.

Lemma 3.1 *Let* $\mathsf{A}\in\mathcal{T}^1(\Gamma)$*,* $a\in\mathcal{C}(\Gamma\times\Gamma)$ *its component function,* μ *a weight on* V *and* c *the coefficient function of the pair* (A,μ)*. Then,*

$$c(x,y)=\frac{1}{2}\int_V\Big[\mu(x)a(x,z,y)+\mu(y)a(y,x,z)-\mu(z)a(z,x,y)\Big]\,dz,$$

for any $x,y\in V$ *with* $x\neq y$*. In addition, for any* $x\in V$ *we get that*

$$\int_V c(x,y)\,dy=\int_V c(y,x)\,dy=\frac{\mu(x)}{2}\int_{V\times V}a(x,y,z)\,dydz+\frac{1}{2}\int_V\mu(z)a(z,x,x)\,dz.$$

In particular, $c(x,y)=0$ *when* $d(x,y)>2$*,* $c(x,y)=-\dfrac{1}{2}\displaystyle\int_V\mu(z)a(z,x,y)\,dz$ *if* $d(x,y)=2$*, whereas* $c(x,y)=\dfrac{1}{2}\displaystyle\int_V\Big[\mu(x)a(x,z,y)+\mu(y)a(y,x,z)\Big]\,dz$ *if* $d(x,y)=1$ *and* x *and* y *do not belong to any triangle.*

Proof. Firstly, if $x\neq y$ applying Identity (7) to $u=\varepsilon_y$ and $v=\varepsilon_x$ we obtain that

$$c(x,y)=-\nu(x)\mathcal{L}(\varepsilon_y)(x)=-\nu(y)\mathcal{L}^*(\varepsilon_x)(y)$$

and hence, applying now Identity (8) to $\mathcal{L}^*$ and $u=\varepsilon_x$, we get that

$$0=\int_V\mathcal{L}^*(\varepsilon_x)\,d\nu=\nu(x)\mathcal{L}^*(\varepsilon_x)(x)-\int_V c(x,y)\,dy=\frac{1}{2}\int_V\langle\mathsf{A}\mathsf{d}\varepsilon_x,\mathsf{d}\varepsilon_x\rangle\,d\mu-\int_V c(x,y)\,dy.$$

In conclusion

$$\int_V c(x,y)\,dy=\frac{1}{2}\int_V\langle\mathsf{A}\mathsf{d}\varepsilon_x,\mathsf{d}\varepsilon_x\rangle\,d\mu=\frac{1}{2}\int_V\langle\mathsf{A}^*\mathsf{d}\varepsilon_x,\mathsf{d}\varepsilon_x\rangle\,d\mu=\int_V c^*(x,y)\,dy=\int_V c(y,x)\,dy.$$

Given $x,y\in V$, let h_x and g_y be the component functions of the vector fields $\mathsf{h}_x=\mathsf{d}\varepsilon_x$ and $\mathsf{g}_y=\mathsf{A}\mathsf{d}\varepsilon_y$, respectively.

On one hand, $h_x(x,z)=-1$, if $z\sim x$, $h_x(y,x)=1$, if $y\sim x$ and $h_x(y,z)=0$, otherwise, which implies that if $\mathsf{f}\in\mathcal{X}(\Gamma)$ then $\langle\mathsf{d}\varepsilon_x,\mathsf{f}\rangle(x)=-\displaystyle\int_V f(x,y)\,dy$, whereas if $y\neq x$, $\langle\mathsf{d}\varepsilon_x,\mathsf{f}\rangle(y)=f(y,x)$

On the other hand, $g_y(w,t)=a(w,t,y)$, if $w\neq y$ and $g_y(y,t)=-\displaystyle\int_V a(y,t,z)\,dz$, which implies that $\langle\mathsf{A}\mathsf{d}\varepsilon_y,\mathsf{d}\varepsilon_x\rangle(z)=g_y(z,x)$ if $z\neq x$ and $\langle\mathsf{A}\mathsf{d}\varepsilon_y,\mathsf{d}\varepsilon_x\rangle(x)=-\displaystyle\int_V g_y(x,t)\,dt$.

Therefore, for any $x, y \in V$ we obtain the identity

$$\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle\, d\mu = -\frac{\mu(x)}{2}\int_V g_y(x,t)\,dt + \frac{1}{2}\int_V \mu(z)\, g_y(z,x)\,dz.$$

So, if $x \neq y$

$$\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle\, d\mu = -\frac{\mu(x)}{2}\int_V a(x,t,y)\,dt - \frac{\mu(y)}{2}\int_V a(y,x,z)\,dz + \frac{1}{2}\int_V \mu(z)\, a(z,x,y)\,dz,$$

whereas

$$\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_x, \mathsf{d}\varepsilon_x\rangle\, d\mu = \frac{\mu(x)}{2}\int_V\int_V a(x,t,z)\,dzdt + \frac{1}{2}\int_V \mu(z)\, a(z,x,x)\,dz.$$

Finally, if $d(x,y) \geq 2$ then $a(x,z,y) = a(y,x,z) = 0$ for any $z \in V$ since $x \not\sim y$, which implies that $c(x,y) = -\frac{1}{2}\int_V \mu(z)a(z,x,y)\,dz$. Moreover when $d(x,y) > 2$ then $a(z,x,y) = 0$, since $x \sim z$ and $y \sim z$, implies that $d(x,y) \leq 2$. In addition, if $d(x,y) = 1$ but x and y are not vertices of any triangle, then there are not $z \in V$ such that $z \sim x, y$, which implies that $a(z,x,y) = 0$ for any $z \in V$ and hence $c(x,y) = \frac{1}{2}\int_V \Big[\mu(x)a(x,z,y) + \mu(y)a(y,x,z)\Big]\,dz$. ∎

We remark that when A is a diagonal field of endomorphisms, then

$$c(x,y) = \frac{1}{2}\Big(\mu(x)a(x,y,y) + \mu(y)a(y,x,x)\Big), \quad \text{for any } x, y \in V$$

and hence $c(x,y) = 0$ when $d(x,y) \neq 1$. In addition, $c = c^*$ since any diagonal field of endomorphisms is symmetric. The above equality proves that, in general, A can not be uniquely determined by c. For instance, if we take $f \in \frac{1}{\mu}\mathcal{C}^s(\Gamma)$, $g \in \frac{1}{\mu}\mathcal{C}^a(\Gamma)$ and A the diagonal field of endomorphism whose coefficient function is given by $a(x,y,y) = f(x,y) + g(x,y)$ for any $x, y \in V$, then $c = f$ and hence c does not depend on g.

Now we can describe explicitly the operator $\mathcal{L}$ and also its associated bilinear form. For that, it will be useful to consider the symmetric and skew-symmetric parts of A, that are the fields of endomorphisms $\mathsf{A}_s, \mathsf{A}_a$ defined respectively by $\mathsf{A}_s(x) = \frac{1}{2}(\mathsf{A}(x) + \mathsf{A}^*(x))$ and $\mathsf{A}_a(x) = \frac{1}{2}(\mathsf{A}(x) - \mathsf{A}^*(x))$ for any $x \in V$. Moreover, the functions $c_s = \frac{1}{2}(c + c^*)$ and $c_a = \frac{1}{2}(c - c^*)$ are the coefficient functions of the pairs (A_s, μ) and (A_a, μ), respectively.

Proposition 3.2 *For any $u \in \mathcal{C}(V)$ we have that*

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)}\int_V c(x,y)\,\Big(u(x) - u(y)\Big)\,dy, \quad \textit{for any } x \in V.$$

Moreover, if $v \in \mathcal{C}_0(V)$, then

$$\int_V v\,\mathcal{L}(u)\,d\nu = \frac{1}{2}\int_{V\times V} c_s(x,y)\,\Big(u(x) - u(y)\Big)\Big(v(x) - v(y)\Big)\,dxdy + \int_{V\times V} c_a(x,y)\,u(x)\,v(y)\,dxdy.$$

Proof. From the definition of the coefficient function we obtain that

$$\mathcal{L}(u)(x) = \frac{-1}{\nu(x)}\int_V c(x,y)\,u(y)\,dy + \frac{u(x)}{2\nu(x)}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_x, \mathsf{d}\varepsilon_x\rangle\,d\mu,$$

for any $u \in \mathcal{C}(V)$ and any $x \in V$. Moreover, in the proof of Lemma 3.1, we have showed that

$$\int_V c(x,y)\,dy = \frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_x, \mathsf{d}\varepsilon_x\rangle\,d\mu,$$

and hence the first equality follows. Now, given $v \in \mathcal{C}_0(V)$, then

$$\int_V v\mathcal{L}(u)\,d\nu = \int_{V\times V} c(x,y)\,v(x)\,\Big(u(x)-u(y)\Big)\,dydx$$

and hence

$$\begin{aligned}\int_V v\mathcal{L}(u)\,d\nu &= \frac{1}{2}\int_{V\times V} c(x,y)\,v(x)\,\Big(u(x)-u(y)\Big)\,dydx\\ &\quad -\frac{1}{2}\int_{V\times V} c(y,x)\,v(y)\,\Big(u(x)-u(y)\Big)\,dxdy\\ &= \frac{1}{2}\int_{V\times V} c(x,y)\,\Big(u(x)-u(y)\Big)\,\Big(v(x)-v(y)\Big)\,dydx\\ &\quad +\int_{V\times V} c_{\mathrm{a}}(x,y)\,v(y)\,\Big(u(x)-u(y)\Big)\,dxdy.\end{aligned}$$

The result follows taking into account that from Lemma 3.1, $\int_V c_{\mathrm{a}}(x,y)\,dx = 0$ for any $x \in V$ and that

$$\int_{V\times V} c(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)dxdy = \int_{V\times V} c_{\mathrm{s}}(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)dxdy. \blacksquare$$

The above proposition implies that the bilinear form $\int_V v\mathcal{L}(u)\,d\nu$ is symmetric, or equivalently that $\mathcal{L}$ is a self-adjoint operator on $\mathcal{C}_0(V)$, iff $c = c^*$. Therefore, the pair (A, μ) is semi-elliptic iff $c = c^*$ and in addition

$$\int_{V\times V} c(x,y)\,\Big(u(x)-u(y)\Big)^2 dxdy \ge 0, \qquad \text{for any } \ u \in \mathcal{C}_0(V).$$

We remark that the above inequality, does not imply the non negativeness of the function c. For instance, if we consider $K_3 = \{x_1, x_2, x_3\}$ the complete graph, $\mu = 1$ and A the diagonal field of endomorphisms whose component function is given by

$$\begin{aligned} &a(x_1,x_2,x_2) = a(x_2,x_1,x_1) = 3, \qquad a(x_2,x_3,x_3) = a(x_3,x_2,x_2) = 2,\\ &a(x_1,x_3,x_3) = a(x_3,x_1,x_1) = -1, \qquad a(x,y,z) = 0, \text{ otherwise,}\end{aligned}$$

then $c(x,y) = a(x,y,y)$, for any $x, y \in K_3$ and the pair (A, μ) is elliptic.

Moreover the fact $c = c^*$ does not imply the symmetry of the field of endomorphisms, or in an equivalent manner $c_{\mathrm{a}} = 0$ does not imply that $\mathsf{A}_{\mathrm{a}} = \mathsf{0}$. For instance, if we consider again $K_3 = \{x_1, x_2, x_3\}$ the complete graph, $\mu = 1$ and A the skew-symmetric field of endomorphisms whose component function is given by

$$a(x_1,x_2,x_3) = -a(x_1,x_3,x_2) = a(x_2,x_1,x_3) = -a(x_2,x_3,x_1) = 1, \qquad a(x,y,z) = 0, \text{ otherwise,}$$

then $c(x,y)=0$, for any $x,y\in K_3$. Therefore, it will be useful to assume that the field A is symmetric when $\mathcal{L}$ is self-adjoint. So, in the sequel both properties will be consider equivalent.

On the other hand, it is clear that if A is a symmetric field and c is a non negative function, then the pair (A,μ) is semi-elliptic. If in addition $c(x,y)>0$ when $x\sim y$, then the pair (A,μ) is elliptic. For this reason , we say that the operator $\mathcal{L}$, or equivalently the pair (A,μ), is *strongly elliptic*, if $c=c^*$, $c\ge 0$ and moreover $c(x,y)>0$ when $x\sim y$.

Observe that when $\mathcal{L}$ is a strongly elliptic operator, then for any finite subset $F\subset V$ the value $c(F)=\frac{1}{2}\min\{c(x,y): x,y\in\bar{F}, x\sim y\}$ satisfies that $c(F)>0$ and hence we get that

$$\int_V u\mathcal{L}(u)\,d\nu \ge c(F)\int_V \langle \mathsf{d}u,\mathsf{d}u\rangle\,dx, \qquad \text{for any} \;\; u\in\mathcal{C}(F).$$

In view of applications, it is of interest to describe when the pair (A,μ) is strongly elliptic. The following result establishes simple conditions on the component function of A to ensure this property, independently of the weight μ.

Lemma 3.3 *Let* A *be a symmetric field of endomorphisms and* $a\in\mathcal{C}(\Gamma\times\Gamma)$ *its component function. If* $a(z,x,y)\le 0$ *for all* $x,y,z\in V$ *with* $x\ne y$ *and* $\displaystyle\int_V a(x,y,z)\,dz\ge 0$ *for all* $x,y\in V$*, then* $c\ge 0$ *for any weight* μ *and* $c=0$ *iff* $\mathsf{A}=0$*. In addition, if when* $x\sim y$ *it is verified that either* $\displaystyle\int_V a(x,y,z)\,dz>0$ *or there exists* $z\in V$ *such that* $a(z,x,y)<0$*, then* $c(x,y)>0$ *for any weight* μ*.*

The above lemma allows us to solve in some cases the problem of identifying the field of endomorphisms from the coefficient function. Specifically, if A_1 and A_2 are fields of endomorphisms such that $\mathsf{A}_1-\mathsf{A}_2$ verifies the hypotheses of the above lemma, then $c_1=c_2$ iff $\mathsf{A}_1=\mathsf{A}_2$, where c_1 and c_2 are the coefficient functions associated to (A_1,μ) and (A_2,μ).

Observe that if for each $x\in V$ we consider the symmetric matrix of order $k(x)$ given by $A(x)=\Big(a(x,y,z)\Big)_{\substack{y\sim x\\ z\sim x}}$, then the hypotheses of the above lemma say nothing else than $A(x)$ is a diagonally dominant M-matrix and hence a positive semi-definite matrix. Now, we generalized the above identification property.

Proposition 3.4 *Let* μ *be a weight and* c_1 *and* c_2 *the coefficient functions of the pairs* (A_1,μ) *and* (A_2,μ)*, respectively. If* $\mathsf{A}_1-\mathsf{A}_2$ *is symmetric and positive semi-definite, then* $c_1=c_2$ *iff* $\mathsf{A}_1=\mathsf{A}_2$*.*

Proof. It suffices to prove that if A is a symmetric and positive definite field of endomorphisms, then $c=0$ iff $\mathsf{A}=0$. Clearly, $c=0$ when $\mathsf{A}=0$. Conversely, if $c=0$ then, from Identity (7)

$$0=\int_V \langle \mathsf{A}\mathsf{d}u,\mathsf{d}v\rangle\,d\mu$$

and hence $\langle \mathsf{A}\mathsf{d}u,\mathsf{d}v\rangle=0$ for any $u,v\in\mathcal{C}(V)$, since A is positive semi-definite. Moreover, if f is the component function of the field $\mathsf{A}\mathsf{d}u$, then for any $x,y\in V$, $x\ne y$, taking $v=\varepsilon_y$, we get that

$$0=\int_V f(x,z)\Big(\varepsilon_y(z)-\varepsilon_y(x)\Big)\,dz=f(x,y).$$

Therefore, $\mathsf{A}\mathsf{d}u = 0$ for any $u \in \mathcal{C}(V)$. If we take $u = \varepsilon_z$, $z \sim x$, then for any $y \sim x$

$$0 = \sum_{w \in V} a(x,y,w)\Big(\varepsilon_z(w) - \varepsilon_z(x)\Big) = a(x,y,z)$$

and hence $\mathsf{A} = \mathsf{0}$. ∎

We conclude this section with some remarks. Firstly, when $\nu = \mu = 1$, the definition of the Laplace operator of a weighted graph is the discrete analogue of the Laplace operator of a differentiable Riemannian manifold, whereas the case $\nu = \mu$ corresponds to the expression of this operator in coordinates, where μ plays the role of the module of the Jacobian determinant. In general, the Laplace operator can also be interpreted as the discrete analogue of the Laplace-Beltrami operator of a *weighted Riemannian manifold*, see for instance [1]. In this context, the discrete operators studied in the literature basically correspond to the case in which the field A is an orthogonal metric and $\mu = 1$. The particular case $\nu = 1$, leads to the so-called *combinatorial Laplacian*, whereas when $\nu(x) = \displaystyle\int_V c(x,y)\, dy$, the corresponding Laplace operator is the so-called *probabilistic Laplacian*, which is associated with a reversible random walk whose stationary distribution is ν. Of course, the above concept can be extended to general metrics as follows: if we suppose that the pair (A, μ) is strongly-elliptic, then we can define the probabilistic Laplacian by considering ν as before. In this case, the associated reversible random walk is not necessarily of nearest neighbor type.

In the electrical network context, if $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ the expression $\mathsf{f} = \mathsf{A}\mathsf{d}u$ can be interpreted as a general linear *Ohm's Law* described in terms of the *admittance field* A, where u represents the *potential*, $\mathsf{d}u$ the *voltage* and f the *current* of the network. Therefore for any *current source* $g \in \mathcal{C}(V)$, the identity $\mathsf{div}\,\mathsf{f} = g$, that is $\mathcal{L}(u) = -g$, represents the state equation of the network, obtained by application of the *Kirchhoff's Laws*, and then c is nothing else than the *conductance function* of the network. For this reason (Γ, μ, ν, c) is called *weighted network*. In the electrical realm one can find many situations that require non-diagonal admittance matrices. This is the case of the so-called *linear multiports* that reflect the existence of devices other than resistances. For instance, the most well known 2-port is the *transformer* that consists in a pair of *coupled inductors* with inductances L_1 and L_2 respectively, and whose mutual inductance is given by $M = k\sqrt{L_1 L_2}$. The parameter k is called *coupling coefficient* and takes is values from 0 to 1. The admittance matrix of this transformer is $\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}$. Another usual example of electrical device is a *gyrator*, that is a 2-port whose admittance matrix is given by $\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}$, where $g = \dfrac{1}{r}$ is named *gyration constant*. For more examples and a deeper analysis of the mathematical treatment of multiports we refer to the interested reader to [16].

4 Integration by Parts and Green's Identities

In this section we aim to establish the discrete analogous of the Integration by Parts technique and moreover we are also interested in some useful consequences of it, namely the Divergence Theorem and the Green's Identities, that play a fundamental role in the analysis of boundary value problems.

These results are given on a finite vertex subset, the discrete equivalent to a compact region, so we need to define the discrete analogous of the exterior normal vector field to the set. Throughout this section we consider fixed (Γ, μ, ν) a weighted graph and $F \subset V$ a fixed proper finite and connected vertex subset.

The *normal vector field to* F is defined as $\mathsf{n}_F = -\mathsf{d}\chi_F$. Therefore, the component function of n_F is given by $n_F(x,y) = 1$ when $y \sim x$ and $(x,y) \in \delta(F^c) \times \delta(F)$, $n_F(x,y) = -1$ when $y \sim x$ and $(x,y) \in \delta(F) \times \delta(F^c)$ and $n_F(x,y) = 0$, otherwise. In consequence, $\mathsf{n}_{F^c} = -\mathsf{n}_F$ and $\mathsf{supp}(\mathsf{n}_F) = \delta(F^c) \cup \delta(F)$.

Proposition 4.1 (Integration by Parts) *Given* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *a* ν*-flow, then for any functions* $u, v \in \mathcal{C}(\bar{F})$ *it is verified that*

$$\int_F v\langle \mathsf{f}, \mathsf{d}u\rangle \, d\nu = \int_{\bar{F}\times\bar{F}} f_F(x,y)v(x)u(y)\nu(x)\,dxdy - \int_F \mathsf{Div}\,(\nu\mathsf{f})\,uv\,dx$$
$$- \int_{\delta(F)} v\langle \mathsf{f}_F, \mathsf{d}u\rangle\,d\nu + \int_{\delta(F)} \langle \mathsf{f}, \mathsf{n}_F\rangle\,uv\,d\nu$$

and therefore,

$$\int_F \Big(v\langle \mathsf{f}, \mathsf{d}u\rangle + u\langle \mathsf{f}, \mathsf{d}v\rangle\Big)\,d\nu = -\,2\int_F \mathsf{Div}\,(\nu\mathsf{f})\,uv\,dx$$
$$- \int_{\delta(F)} \Big(v\langle \mathsf{f}_F, \mathsf{d}u\rangle + u\langle \mathsf{f}_F, \mathsf{d}v\rangle\Big)\,d\nu + 2\int_{\delta(F)} \langle \mathsf{f}, \mathsf{n}_F\rangle\,uv\,d\nu.$$

Proof. Firstly, taking into account that $\mathsf{f} = \mathsf{f}_F$ on F we get that

$$\int_{\bar{F}} v\langle \mathsf{f}_F, \mathsf{d}u\rangle\,d\nu = \int_F v\langle \mathsf{f}, \mathsf{d}u\rangle\,d\nu + \int_{\delta(F)} v\langle \mathsf{f}_F, \mathsf{d}u\rangle\,d\nu.$$

On the other hand, $\mathsf{Div}\,(\nu\mathsf{f}_F)(x) = \nu(x)\displaystyle\int_{\bar{F}} f_F(x,y)\,dy$ for any $x \in V$, since f is a ν-flow, and hence

$$\int_{\bar{F}} v\langle \mathsf{f}_F, \mathsf{d}u\rangle\,d\nu = \int_{\bar{F}\times\bar{F}} \nu(x)f_F(x,y)u(y)v(x)\,dydx - \int_{\bar{F}} \mathsf{Div}\,(\nu\mathsf{f}_F)\,uv\,dx.$$

So, the first claim is consequence of the identities $\mathsf{Div}\,(\nu\mathsf{f}_F) = \mathsf{Div}\,(\nu\mathsf{f})$ on F and $\mathsf{Div}\,(\nu\mathsf{f}_F) = -\langle \mathsf{f}, \mathsf{n}_F\rangle\nu$ on $\delta(F)$.

Finally, the second claim is a standard consequence of the first one taking into account that

$$\int_{\bar{F}\times\bar{F}} f_F(x,y)u(x)v(y)\nu(x)\,dxdy = -\int_{\bar{F}\times\bar{F}} f_F(x,y)v(x)u(y)\nu(x)\,dxdy,$$

since f is a ν-flow. ∎

Corollary 4.2 (Divergence Theorem) *For any* μ*-flow* $\mathsf{g} \in \mathcal{X}(\Gamma)$, *it is verified that*

$$\int_F \mathsf{div}\,\mathsf{g}\,d\nu = \int_{\delta(F)} \langle \mathsf{g}, \mathsf{n}_F\rangle\,d\mu.$$

Proof. If $\mathsf{f} = \frac{\mu}{\nu}\,\mathsf{g}$, then f is a ν-flow and hence $\dfrac{1}{\nu}\mathsf{Div}\,(\nu \mathsf{f}) = \dfrac{1}{\nu}\mathsf{Div}\,(\mu \mathsf{g}) = \mathsf{div}\,\mathsf{g}$. The result follows taking $u = v = \chi_{\bar{F}}$ in the second identity of Proposition 4.1. ∎

Note that when $\mu = 1$ and $\nu = k$ the equality in the above corollary coincides with the one obtained in [17].

Our next objective is to describe the discrete version of Green's Identities on F, for the second order operator $\mathcal{L}(u) = -\mathsf{div}\,(\mathsf{A}du)$, where A is a symmetric field of endomorphisms. For any $u \in \mathcal{C}(\bar{F})$, from the first equality of Proposition 3.2 we get that

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_{\bar{F}} c(x,y)\Big(u(x) - u(y)\Big)\, dy + q_{\mathsf{A}}(x)u(x), \qquad x \in F, \tag{10}$$

where $q_{\mathsf{A}}\colon F \longrightarrow \mathbb{R}$ is defined as

$$q_{\mathsf{A}}(x) = \frac{1}{\nu(x)} \int_{\delta(\bar{F})} c(x,y)\, dy = -\frac{1}{2\nu(x)} \int_{\delta(\bar{F})\times\delta(F)} \mu(z)a(z,x,y)\, dzdy, \qquad x \in F. \tag{11}$$

Note that $\mathsf{supp}(q_{\mathsf{A}}) \subset \delta(F^c)$ and $q_{\mathsf{A}} = 0$ when A is a diagonal field of endomorphisms.

Identity (10) shows that for any $u \in \mathcal{C}(\bar{F})$ the values of $\mathcal{L}(u)$ on F appear as the sum of two terms of different nature: The first one, $\dfrac{1}{\nu(x)} \displaystyle\int_{\bar{F}} c(x,y)\Big(u(x) - u(y)\Big)\, dy$, that we call *the principal part of $\mathcal{L}$ on F*, looks like a combinatorial laplacian and it depends on the connectivity between vertices in F as well as on the connectivity between vertices in F and in $\delta(F)$. The second one, $q_{\mathsf{A}}u$, is a 0-order term that represents the kind of connectivity between F and its *exterior*, $(\bar{F})^c$. In other words, the operator $\mathcal{L}$ on $\mathcal{C}(\bar{F})$ is a *combinatorial Schrödinger operator* whose *ground state* is q_{A}, see [3].

To develop a discrete version of the Green's Identities it is also necessary to introduce a discrete analogue of the co-normal derivative for functions supported by $\bar{F}$. So, fixed F, for any field of endomorphisms A, we define the *co-normal derivative on F with respect to* A as the linear operator $\dfrac{\partial}{\partial \mathsf{n}_{\mathsf{A}}}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any $u \in \mathcal{C}(\bar{F})$ the function given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\right)(x) = \frac{1}{\nu(x)} \int_{F} c(x,y)\,\Big(u(x) - u(y)\Big)\, dy, \qquad x \in \delta(F). \tag{12}$$

When A is a diagonal field of endomorphisms the above definition coincides with the given by other authors see for instance [3, 5, 7] and references therein.

Proposition 4.3 (Green's Identities) *For any $u, v \in \mathcal{C}(\bar{F})$ the following identities hold:*

(i) First Green Identity

$$\int_F v\mathcal{L}(u)\, d\nu = \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_F(x,y)\Big(u(x) - u(y)\Big)\Big(v(x) - v(y)\Big)dxdy + \int_F q_{\mathsf{A}} uv\, d\nu - \int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\, d\nu.$$

(ii) Second Green Identity

$$\int_F \Big(v\mathcal{L}(u) - u\mathcal{L}(v)\Big)\, d\nu = \int_{\delta(F)} \left(u\frac{\partial v}{\partial \mathsf{n}_{\mathsf{A}}} - v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\right)\, d\nu.$$

Proof. Tacking into account the expression (10), we obtain that for any $u,v\in\mathcal{C}(\bar F)$

$$\int_F v\,\mathcal{L}(u)\,d\nu=\int_{F\times\bar F}c(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx+\int_F q_{\mathsf{A}}uv\,d\nu$$
$$=\int_{\bar F\times\bar F}c_{F}(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx+\int_F q_{\mathsf{A}}uv\,d\nu-\int_{\delta(F)}v\,\frac{\partial u}{\partial\mathsf{n}_{\mathsf{A}}}\,d\nu,$$

and the First Green Identity follows reasoning as in the proof of Proposition 3.2 to get

$$\int_{\bar F\times\bar F}c_F(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx=\frac{1}{2}\int_{\bar F\times\bar F}c_F(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dydx.$$

The Second Green Identity is a direct consequence of the first one. ∎

When $\mu=1$ and A is a diagonal field, the above Green's Identities correspond to those obtained by several authors, see principally [3, 5, 7, 10, 14].

5 Boundary Value Problems on weighted networks

Our aim in this section is to describe boundary value problems on a finite vertex subset of a weighted graph, or network, associated with second order operators on $\mathcal{C}(V)$ as well as to provide its variational or weak formulation. We will newly suppose fixed the weighted graph (Γ,μ,ν), a finite subset $F\subset V$, a symmetric field of endomorphisms A, a vector field f and two vertex functions $q\in\mathcal{C}(F)$ and $p\in\mathcal{C}(\delta(F))$. In addition, we also consider c the coefficient function of the pair (A,μ).

Associated with the above data, we define the *difference operator* $\mathcal{L}:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ given by

$$\mathcal{L}(u)=-\mathsf{div}\,(\mathsf{A}\mathsf{d}u)+\langle\mathsf{f},\mathsf{d}u\rangle+q\,u \tag{13}$$

and also the *boundary operator* $\mathcal{U}:\mathcal{C}(\bar F)\longrightarrow\mathcal{C}(\delta(F))$ given by

$$\mathcal{U}(u)=\frac{\partial u}{\partial\mathsf{n}_{\mathsf{A}}}+\langle\mathsf{f}_{F},\mathsf{d}u\rangle+p\,u. \tag{14}$$

If $a\in\mathcal{C}(\Gamma\times\Gamma)$ and $f\in\mathcal{C}(\Gamma)$ are the component functions of A and f, respectively, consider $\tilde a\in\mathcal{C}(\Gamma\times\Gamma)$ defined for any $x,y,z\in V$ as $\tilde a(x,y,z)=a(x,y,z)$, when $z\neq y$ and $\tilde a(x,y,y)=a(x,y,y)-\dfrac{1}{2\mu(x)}\Big(\nu(x)f(x,y)+\nu(y)f(y,x)\Big)$. Then, if $\tilde{\mathsf{A}}$ is the field of endomorphisms whose coefficient function is $\tilde a$ and $\tilde{\mathsf{f}}=\frac{1}{\nu}\,\widehat{(\nu\mathsf{f})}$ it is easy to check that

$$\mathcal{L}(u)=-\mathsf{div}\,(\tilde{\mathsf{A}}\mathsf{d}u)+\langle\tilde{\mathsf{f}},\mathsf{d}u\rangle+qu\;\text{ and }\;\mathcal{U}(u)=\frac{\partial u}{\partial\mathsf{n}_{\tilde{\mathsf{A}}}}+\langle\tilde{\mathsf{f}}_{F},\mathsf{d}u\rangle+p\,u,$$

for any $u\in\mathcal{C}(V)$. Since $\tilde{\mathsf{A}}$ is a symmetric field of endomorphisms and $\tilde{\mathsf{f}}$ is a ν-flow, we can suppose without loss of generality that the fixed field f is a ν-flow.

Given $\delta(F)=H_1\cup H_2$ a partition of $\delta(F)$ and functions $g\in\mathcal{C}(F)$, $g_1\in\mathcal{C}(H_1)$, $g_2\in\mathcal{C}(H_2)$, a *boundary value problem on* F consists in finding $u\in\mathcal{C}(\bar F)$ such that

$$\mathcal{L}(u)=g\;\text{ on }\;F,\quad\mathcal{U}(u)=g_1\;\text{ on }\;H_1\quad\text{and}\quad u=g_2\;\text{ on }\;H_2. \tag{15}$$

The associated homogeneous boundary value problem consists in finding $u \in \mathcal{C}(\bar{F})$ such that $\mathcal{L}(u) = 0$ on F, $\mathcal{U}(u) = 0$ on H_1 and $u = 0$ on H_2. It is clear that the set of solutions of the homogeneous boundary value problem is a vector subspace of $\mathcal{C}(F \cup H_1)$ that we denote by $\mathcal{V}$. Moreover if Problem (15) has solution and u is a particular one, then $u + \mathcal{V}$ describes the set of all its solutions.

Problem (15) is generically known as a *mixed Dirichlet-Robin problem*, specially when $p \neq 0$, and $H_1, H_2 \neq \emptyset$, and summarizes the different boundary value problems that appear in the literature with the following proper names:

i) *Dirichlet problem*: $\emptyset \neq H_2 = \delta(F)$ and hence $H_1 = \emptyset$.

ii) *Robin problem*: $p \neq 0$, $\emptyset \neq H_1 = \delta(F)$ and hence $H_2 = \emptyset$.

iii) *Neumann problem*: $p = 0$, $\emptyset \neq H_1 = \delta(F)$ and hence $H_2 = \emptyset$.

iv) *Mixed Dirichlet-Neumann problem*: $p = 0$ and $H_1, H_2 \neq \emptyset$.

In addition, when Γ is a finite graph it is possible that $H_1 = H_2 = \emptyset$ and hence $F = V$. In this case, Problem (15) is known as the *Poisson equation* on V.

Consider now the difference operator $\mathcal{L}^*: \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ given by

$$\mathcal{L}^*(u) = -\mathsf{div}\,(\mathsf{A}\mathsf{d}u) - \langle \mathsf{f}, \mathsf{d}u \rangle + \Big(q - \frac{2}{\nu}\,\mathsf{Div}\,(\nu \mathsf{f}) \Big)\, u, \tag{16}$$

the boundary operator $\mathcal{U}^*: \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ given by

$$\mathcal{U}^*(u) = \frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}} - \langle \mathsf{f}_{F}, \mathsf{d}u \rangle + \Big(p + 2\langle \mathsf{f}, \mathsf{n}_{F} \rangle \Big)\, u \tag{17}$$

and the (homogeneous) boundary value problem on F

$$\mathcal{L}^*(u) = 0 \;\text{ on }\; F, \quad \mathcal{U}^*(u) = 0 \;\text{ on }\; H_1 \quad\text{ and }\quad u = 0 \;\text{ on }\; H_2. \tag{18}$$

The above problem is called *the Adjoint Problem of (15)* and the subspace of its solutions is denoted by $\mathcal{V}^*$. Moreover, we say that Problem (15) is *self-adjoint* when $\mathcal{L} = \mathcal{L}^*$ on F and $\mathcal{U} = \mathcal{U}^*$ on H_1. This property implies that $\mathsf{Div}\,(\nu \mathsf{f}) = 0$ on F and that $\langle \mathsf{f}, \mathsf{n}_{F} \rangle = 0$ on H_1. In particular problem (15) is self-adjoint when $\mathsf{f} = \mathsf{0}$ on $\bar{F}$.

To describe the conditions that assure the existence and uniqueness of solutions of the boundary value problem (15) we need to extend the Second's Green Identity to operators $\mathcal{L}$ and $\mathcal{L}^*$.

Proposition 5.1 *For any $u, v \in \mathcal{C}(\bar{F})$ it is verified that*

$$\int_F \Big(v\mathcal{L}(u) - u\mathcal{L}^*(v) \Big)\, \nu\, dx = \int_{\delta(F)} \Big(u\mathcal{U}^*(v) - v\mathcal{U}(u) \Big)\, \nu\, dx.$$

In particular, problems (15) and (18) are mutually adjoint.

Proof. The first claim is a direct consequence of the Integration by Parts and the Second's Green Identity. Moreover, if $u, v \in \mathcal{C}(F \cup H_1)$ are such that $\mathcal{U}(u) = \mathcal{U}^*(v) = 0$ on H_1, then $\displaystyle\int_{\delta(F)} \Big(u\mathcal{U}^*(v) - v\mathcal{U}(u)\Big)\, \nu\, dx = 0$ and hence $\displaystyle\int_F v\, \mathcal{L}(u)\, \nu\, dx = \int_F u\, \mathcal{L}^*(v)\, \nu\, dx$; that is, problems (15) and (18) are mutually adjoint. ∎

Proposition 5.2 (Fredholm Alternative) *Given $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$, the boundary value problem*

$$\mathcal{L}(u) = g \ \text{ on } \ F, \quad \mathcal{U}(u) = g_1 \ \text{ on } \ H_1 \ \text{ and } \ u = g_2 \ \text{ on } \ H_2$$

has solution iff

$$\int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu + \int_{H_2} g_2 \langle \mathsf{f}_F, \mathsf{d}v\rangle\, d\nu = \int_{H_2} g_2 \frac{\partial v}{\partial \mathsf{n}_{\mathsf{A}}}\, d\nu, \qquad \text{for each } v \in \mathcal{V}^*.$$

In addition, when the above condition holds, then there exists a unique solution $u \in \mathcal{C}(\bar{F})$ of the boundary value problem such that $\displaystyle\int_{\bar{F}} uv\, d\nu = 0$, for any $v \in \mathcal{V}^*$.

Proof. First, observe that problem (15) is equivalent to the boundary value problem

$$\mathcal{L}(u) = g - \mathcal{L}(g_2) \ \text{ on } \ F, \ \ \mathcal{U}(u) = g_1 - \mathcal{U}(g_2) \ \text{ on } \ H_1 \ \text{ and } \ u = 0 \ \text{ on } \ H_2$$

in the sense that u is a solution of this problem iff $u + g_2$ is a solution of (15).

Consider now the linear operators $\mathcal{F}, \mathcal{F}^*\colon \mathcal{C}(F \cup H_1) \longrightarrow \mathcal{C}(F \cup H_1)$ defined as

$$\mathcal{F}(u) = \begin{cases} \mathcal{L}(u) & \text{on } F, \\ \mathcal{U}(u) & \text{on } H_1 \end{cases} \qquad \text{and} \qquad \mathcal{F}^*(u) = \begin{cases} \mathcal{L}^*(u) & \text{on } F, \\ \mathcal{U}^*(u) & \text{on } H_1, \end{cases}$$

respectively. Then, $\ker\mathcal{F} = \mathcal{V}$, $\ker\mathcal{F}^* = \mathcal{V}^*$ and moreover, by applying Proposition 5.1 for any $u, v \in \mathcal{C}(F \cup H_1)$ it is verified that

$$\begin{aligned} \int_{F \cup H_1} v\mathcal{F}(u)\, d\nu &= \int_F v\mathcal{L}(u)\, d\nu + \int_{\delta(F)} v\mathcal{U}(u)\, d\nu \\ &= \int_F u\mathcal{L}^*(v)\, d\nu + \int_{\delta(F)} u\mathcal{U}^*(v)\, d\nu = \int_{F \cup H_1} u\mathcal{F}^*(v)\, d\nu. \end{aligned}$$

Therefore the operators $\mathcal{F}$ and $\mathcal{F}^*$ are mutually adjoint with respect to the inner product induced in $\mathcal{C}(F \cup H_1)$ by the weight ν and hence $\mathrm{Img}\mathcal{F} = {\mathcal{V}^*}^{\perp}$ by applying the classical Fredholm Alternative. Consequently problem (15) has a solution iff function $\tilde{g} \in \mathcal{C}(F \cup H_1)$ given by $\tilde{g} = g - \mathcal{L}(g_2)$ on F and $\tilde{g} = g_1 - \mathcal{U}(g_2)$ on H_1 verifies that

$$\begin{aligned} 0 = \int_{F \cup H_1} \tilde{g} v\, d\nu &= \int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu - \int_F v\mathcal{L}(g_2)\, d\nu - \int_{H_1} v\mathcal{U}(g_2)\, d\nu \\ &= \int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu - \int_F g_2\mathcal{L}^*(v)\, d\nu - \int_{\delta(F)} g_2\mathcal{U}^*(v)\, d\nu \\ &= \int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu - \int_{H_2} g_2\mathcal{U}^*(v)\, d\nu, \end{aligned}$$

for any $v \in \mathcal{V}^*$. The result follows keeping in mind that $\mathcal{U}^*(v) = \dfrac{\partial v}{\partial \mathsf{n}_{\mathsf{A}}} - \langle \mathsf{f}_{F}, \mathsf{d}v\rangle$ on H_2, for any $v \in \mathcal{C}(F \cup H_1)$. Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition is attained there exists a unique $w \in \mathcal{V}^{*\perp}$ such that $\mathcal{F}(w) = \tilde{g}$. Therefore, $u = w + g_2$ is the unique solution of Problem (15) such that for any $v \in \mathcal{V}^*$

$$\int_{\bar{F}} uv\, d\nu = \int_{F\cup H_1} uv\, d\nu = \int_{F\cup H_1} wv\, d\nu = 0,$$

since $v = 0$ on H_2 and $g_2 = 0$ on $F \cup H_1$. ∎

Observe that as a by-product of the above proof, we obtain that $\dim\mathcal{V} = \dim\mathcal{V}^*$ and hence we can conclude that uniqueness is equivalent to existence for any data.

Next, we establish the variational formulation of the boundary value problem (15), that represents the discrete version of the weak formulation for boundary value problems. In particular, we show that the boundary operators *naturally* associated with the difference operator $\mathcal{L}$ are precisely those of the form $\mathcal{U}(u) = \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}} + \langle \mathsf{f}_{F}, \mathsf{d}u\rangle + p\, u$.

Prior to describe the claimed formulation, we give some useful definitions. *The bilinear form* associated with the boundary value problem (15) is $\mathcal{B}: \mathcal{C}(\bar{F}) \times \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by

$$\mathcal{B}(u,v) = \int_F v\mathcal{L}(u)\, d\nu + \int_{\delta(F)} v\mathcal{U}(u)\, d\nu, \tag{19}$$

and hence, from Proposition 5.1, $\mathcal{B}^*(u,v) = \mathcal{B}(v,u)$ for any $u, v \in \mathcal{C}(\bar{F})$, describes the bilinear form corresponding to the adjoint problem (18). Therefore, Problem (15) is self-adjoint iff $\mathcal{B}$ is symmetric and this occurs iff $\mathsf{f}_{F} = \mathsf{0}$, since applying the Green's Identities and the Integration by Parts Formulae, we obtain that

$$\begin{aligned}\mathcal{B}(u,v) = &\frac{1}{2}\int_{\bar{F}\times\bar{F}} c_F(x,y)\Big(u(x) - u(y)\Big)\Big(v(x) - v(y)\Big)\, dxdy \\ &+ \int_{\bar{F}\times\bar{F}} f_F(x,y)\, u(y)v(x)\nu(x)\, dxdy - \int_F \mathsf{Div}\,(\nu\mathsf{f})\, uv\, dx \\ &+ \int_F (q + q_{\mathsf{A}})\, uv\, d\nu + \int_{\delta(F)} \Big(p + \langle \mathsf{f}, \mathsf{n}_{F}\rangle\Big) uv\, d\nu.\end{aligned} \tag{20}$$

Associated with any pair of functions $g \in \mathcal{C}(F)$ and $g_1 \in \mathcal{C}(H_1)$ we define the linear functional $\ell: \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ as $\ell(v) = \displaystyle\int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu$, whereas for any function $g_2 \in \mathcal{C}(H_2)$ we consider the convex set $K_{g_2} = g_2 + \mathcal{C}(F \cup H_1)$.

Proposition 5.3 (Variational Formulation) *Given $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$, then $u \in K_{g_2}$ is a solution of Problem (15) iff*

$$\mathcal{B}(u,v) = \ell(v), \quad \textit{for any } \; v \in \mathcal{C}(F \cup H_1)$$

and in this case, the set $u + \Big\{w \in \mathcal{C}(F \cup H_1) : \mathcal{B}(w,v) = 0, \text{ for any } v \in \mathcal{C}(F \cup H_1)\Big\}$ describes all solutions of (15).

Proof. A function $u \in K_{g_2}$ satisfies that $\mathcal{B}(u,v) = \ell(v)$ for any $v \in \mathcal{C}(F \cup H_1)$ iff

$$\int_F v(\mathcal{L}(u) - g)\, d\nu + \int_{H_1} v(\mathcal{U}(u) - g_1)\, d\nu = 0.$$

Then, the first result follows by taking $v = \varepsilon_x$, $x \in F \cup H_1$. Finally, $u^* \in K_{g_2}$ is another solution of (15) iff $\mathcal{B}(u^*,v) = \ell(v)$ for any $v \in \mathcal{C}(F \cup H_1)$ and hence iff $\mathcal{B}(u - u^*, v) = 0$ for any $v \in \mathcal{C}(F \cup H_1)$. ∎

Observe that the equality $\mathcal{B}(u,v) = \ell(v)$ for any $v \in \mathcal{C}(F \cup H_1)$ assures that the condition of existence of solution given by the Fredholm Alternative holds, since for any $v \in \mathcal{C}(\bar{F})$ it is verified that

$$\int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu = \mathcal{B}(u,v) = \mathcal{B}^*(v,u) = \int_F u\mathcal{L}^*(v)\, d\nu + \int_{\delta(F)} u\mathcal{U}^*(v)\, d\nu.$$

In particular if $v \in \mathcal{V}^*$ we get that

$$\int_F gv\, d\nu + \int_{H_1} g_1 v\, d\nu = \int_{H_2} u\mathcal{U}^*(v)\, d\nu.$$

On the other hand, we note that the vector subspace

$$\Big\{ w \in \mathcal{C}(F \cup H_1) : \mathcal{B}(w,v) = 0, \text{ for any } v \in \mathcal{C}(F \cup H_1) \Big\}$$

is precisely the set of solutions of the homogeneous boundary value problem associated with (15). So, Problem (15) has solution for any data g, g_1 and g_2 iff it has a unique solution and this occurs iff $w = 0$ is the unique function in $\mathcal{C}(F \cup H_1)$ such that $\mathcal{B}(w,v) = 0$, for any $v \in \mathcal{C}(F \cup H_1)$. Therefore, to assure the existence (and hence the uniqueness) of solutions of Problem (15) for any data it suffices to provide conditions under which $\mathcal{B}(w,w) = 0$ with $w \in \mathcal{C}(F \cup H_1)$, implies that $w = 0$. In particular, this occurs when $\mathcal{B}$ is positive definite on $\mathcal{C}(F \cup H_1)$.

We define *the quadratic form* associated with the boundary value problem (15) as the function $\mathcal{Q}: \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by $\mathcal{Q}(u) = \mathcal{B}(u,u)$; that is,

$$\begin{aligned}\mathcal{Q}(u) = \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_{_F}(x,y)\Big(u(x) - u(y)\Big)^2 dxdy - \int_F \mathsf{Div}\,(\nu\mathsf{f})\, u^2 dx \\ + \int_F (q + q_{\mathsf{A}})\, u^2 d\nu + \int_{\delta(F)} \Big(p + \langle \mathsf{f}, \mathsf{n}_{_F}\rangle\Big) u^2 d\nu.\end{aligned} \tag{21}$$

Our next objective is to establish the conditions under which $\mathcal{Q}$ is positive definite on $\mathcal{C}(F \cup H_1)$. This problem was analyzed in [3, 4], when $\mathsf{f} = 0$ and A is a diagonal field and we adapt here the fundamental results for the general case. Consider a new network $(\bar{\Gamma}(F), \bar{c})$ whose vertex set is $\bar{F}$ and whose conductance is given by $c_{_F}$. In addition, we suppose that the pair (A, μ) is *strongly elliptic on* $\bar{F}$; that is, that $c_{_F} \geq 0$ and $c_{_F}(x,y) > 0$ for any $(x,y) \in (\bar{F} \times \bar{F}) \setminus (\delta(F) \times \delta(F))$ such that $x \sim y$. Then, we consider the operator $\mathcal{L}_{_F}: \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\bar{F})$ given for any $x \in \bar{F}$ by

$$\mathcal{L}_{_F}(u)(x) = \int_{\bar{F}} c_{_F}(x,y)\Big(u(x) - u(y)\Big)\, dy \tag{22}$$

and the function $\rho: \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ defined as $\rho(x) = \nu(x)\Big(q(x) + q_{\mathsf{A}}(x)\Big) - \mathsf{Div}\,(\nu\mathsf{f})(x)$ for $x \in F$ and as $\rho(x) = \nu(x)\Big(p(x) + \langle \mathsf{f}, \mathsf{n}_{_F}\rangle(x)\Big)$ for $x \in \delta(F)$. Then, $\mathcal{L}_{_F} + \rho$ is a Schrödinger operator on $\mathcal{C}(\bar{F})$ whose associated bilinear form is precisely $\mathcal{Q}$. Moreover, $\mathcal{L}_F(u) = -\nu \mathsf{div}\,(\mathsf{A}du)$ on F and $\mathcal{L}_F(u) = \nu\dfrac{\partial u}{\partial \mathsf{n}_F}$ on $\delta(F)$.

If $\sigma \in \mathcal{C}(\bar{F})$ is a *weight on* $\bar{F}$; that is, $\sigma(x) > 0$ for any $x \in \bar{F}$, the function $\rho_\sigma = -\frac{1}{\sigma}\mathcal{L}_F(\sigma)$ takes positive and negative values and moreover $\rho_\sigma(x) > -\displaystyle\int_{\bar{F}} c_F(x,y)\,dy$ for any $x \in \bar{F}$.

Lemma 5.4 [3, Proposition 3.3] *The Schrödinger operator $\mathcal{L}_F + \rho$ is positive semi-definite iff there exists a weight on $\bar{F}$, σ, such that $\rho \geq \rho_\sigma$. In addition, when this condition holds, then $u \in \mathcal{C}(\bar{F})$ verifies $\mathcal{L}_F(u) + \rho u = 0$ iff u is a multiple of σ when $\rho = \rho_\sigma$ and iff $u = 0$, otherwise.*

Now we are ready to establish the fundamental existence results. Until the end of this section, we will assume the following hypotheses:

- H1: The pair (A, μ) is strongly elliptic on $\bar{F}$.
- H2: There exists σ a weight on $\bar{F}$ verifying $q \geq -q_{\mathsf{A}} + \dfrac{1}{\nu}\Big(\rho_\sigma + \mathsf{Div}\,(\nu \mathsf{f})\Big)$ on F and $p \geq \dfrac{\rho_\sigma}{\nu} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_1.

Observe that when the weight σ is constant, then $\rho_\sigma = 0$ and hence hypothesis H2 says nothing else than functions $\nu(q + q_A) - \mathsf{Div}\,(\nu \mathsf{f})$ and $p + \langle \mathsf{f}, \mathsf{n}_F\rangle$ are non-negative on F and on H_1, respectively.

Proposition 5.5 *Suppose that it is not simultaneously satisfied that $q = -q_{\mathsf{A}} + \dfrac{1}{\nu}\Big(\rho_\sigma + \mathsf{Div}\,(\nu \mathsf{f})\Big)$ on F, $p = \dfrac{\rho_\sigma}{\nu} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_1 and $H_2 = \emptyset$. Then for any data $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$ the boundary value problem (15) has a unique solution.*

Proof. If we consider the function $\hat{p} \in \mathcal{C}(\delta(F))$ defined as $\hat{p} = p$ on H_1 and $\hat{p} = \rho_\sigma - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_2, and the quadratic form

$$\begin{aligned}\widehat{\mathcal{Q}}(u) = &\frac{1}{2}\int_{\bar{F}\times\bar{F}} c_F(x,y)\,\Big(u(x) - u(y)\Big)^2 dxdy - \int_F \mathsf{Div}\,(\nu \mathsf{f})\,u^2 dx \\ &+ \int_F (q + q_{\mathsf{A}})\,u^2 d\nu + \int_{\delta(F)} \Big(\hat{p} + \langle \mathsf{f}, \mathsf{n}_F\rangle\Big) u^2 d\nu,\end{aligned}$$

then $\mathcal{Q}(u) = \widehat{\mathcal{Q}}(u)$ for any $u \in \mathcal{C}(F \cup H_1)$. Moreover, Lemma 5.4 assures that under hypotheses H1 and H2, $\widehat{Q}$ is positive definite on $\mathcal{C}(\bar{F})$ and hence $\mathcal{Q}$ is positive definite on $\mathcal{C}(F \cup H_1)$. ∎

Proposition 5.6 *Suppose that $H_2 = \emptyset$, $q = -q_{\mathsf{A}} + \dfrac{1}{\nu}\Big(\rho_\sigma + \mathsf{Div}\,(\nu \mathsf{f})\Big)$ on F and $p = \dfrac{\rho_\sigma}{\nu} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on $\delta(F)$. Then for any data $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(\delta(F))$, the boundary value problem (15) has solution iff it is verified that $\displaystyle\int_F g\,\sigma\,d\nu + \int_{\delta(F)} g_1 \sigma\,d\nu = 0$. Moreover, the solution is unique up to a multiple of σ and there exists a unique solution $u \in \bar{F}$ such that $\displaystyle\int_{\bar{F}} u\,\sigma\,d\nu = 0$.*

Proof. The hypotheses imply that if $v \in \mathcal{C}(\bar{F})$ verifies $\mathcal{Q}(v) = 0$ then v must be a multiple of σ. On the other hand, $\mathcal{V}^* = \{a\sigma : a \in \mathbb{R}\}$ since $\mathcal{Q}$ is also the quadratic form associated to the adjoint problem. Therefore, the conclusions are consequence of the Fredholm Alternative. ∎

When Problem (15) is self-adjoint; that is when $\mathsf{f}_{_F} = 0$, and hypotheses H1 and H2 are in force, we can characterize the solutions of (15) by means of the discrete version of the celebrated *Dirichlet Principle*. Recall that when problem (15) is self-adjoint then the bilinear form $\mathcal{B}$ is symmetric and its associated quadratic functional is given by

$$\mathcal{Q}(u) = \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_{_F}(x,y)\left(u(x)-u(y)\right)^2 dx dy + \int_F (q+q_{\mathsf{A}})\, u^2 d\nu + \int_{\delta(F)} pu^2 d\nu.$$

In addition when $\mathcal{Q}$ is positive definite on $\mathcal{C}(F \cup H_1)$, then for any $w \in \mathcal{C}(F \cup H_1)$, the equality $\mathcal{Q}(w) = 0$ is equivalent to the equality $\mathcal{B}(w,v) = 0$, for any $v \in \mathcal{C}(F \cup H_1)$.

Note that when (15) is self-adjoint, then $\mathsf{Div}\,(\nu\mathsf{f}) = 0$ and $\langle \mathsf{f}, \mathsf{n}_{_F}\rangle = 0$ and hence hypothesis H2 simply says that $q \geq \dfrac{\rho\sigma}{\nu} - q_{_A}$ on F and $p \geq \dfrac{\rho\sigma}{\nu}$ on H_1.

Corollary 5.7 (Dirichlet Principle) *Let $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$ and consider the quadratic functional $\mathcal{J}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by*

$$\mathcal{J}(u) = \mathcal{Q}(u) - 2\ell(u).$$

If Problem (15) is self-adjoint, then $u \in K_{g_2}$ is a solution of problem (15) iff it minimizes $\mathcal{J}$ on $K_{_{g_2}}$.

Proof. It suffices to note that the variational equality in the above proposition is in fact the *Euler identity* for the quadratic functional $\mathcal{Q}$. ∎

We now apply the Dirichlet Principle to a generalization of the problem of identification considered in [5, 7] in which we assume that $H_2 \neq \emptyset$. We remark that the problem considered in the above mentioned works constitutes the discrete counterpart of the inverse continuous conductivity problem for isotropic conductivities, since the considered fields of endomorphisms are diagonal. The problem considered in the next proposition will correspond to the continuous anisotropic conductivities case. This result together with Proposition 3.4 gives a partial answer to the identification problem. We must observe that even in the continuous case for $n \geq 3$, the uniqueness of the conductivities is an open problem for which only partial results have been stated. It is well-known that the inverse anisotropic conductivity problem is not uniquely solvable, since a diffeomorphism which keeps fixed the boundary of the domain produces the same *Dirichlet-to-Neumann map*, see [15]. Therefore, a complete analysis of this problem in the discrete setting will require an adequate extension of the concept of *push-forward* of a field of endomorphisms.

Proposition 5.8 *Let $\mathsf{A}_1, \mathsf{A}_2$ be two symmetric fields of endomorphisms such that the pairs (A_1, μ) and (A_2, μ) are strongly elliptic on $\bar{F}$ and consider also the functions $q_1, q_2 \in \mathcal{C}(F)$ and $p_1, p_2 \in \mathcal{C}(H_1)$. Suppose that $c_F^1 \geq c_F^2$, where c^i is the coefficient function of (A_i, μ), $i = 1, 2$, and that there exists a weight on $\bar{F}$, σ, such that $q_2 + q_{\mathsf{A}_2} \geq q_1 + q_{\mathsf{A}_1} \geq \dfrac{\rho\sigma}{\nu}$ and $p_2 \geq p_1 \geq \dfrac{\rho\sigma}{\nu}$.*

Let functions $u_1, u_2 \in \mathcal{C}(\bar{F})$ *such that* $\mathsf{div}\,(\mathsf{A}_1\mathsf{d}u_1) - q_1 u_1 = \mathsf{div}\,(\mathsf{A}_2\mathsf{d}u_2) - q_2 u_2 = 0$ *on* F, $\dfrac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}} + p_1 u_1 = \dfrac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}} + p_2 u_2 = 0$ *on* H_1 *and* $u_1 = u_2$, $\dfrac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}} = \dfrac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}}$ *on* H_2. *Then,* $u_1 = u_2$ *on* $\bar{F}$, $q_2(x) + q_{\mathsf{A}_2}(x) = q_1(x) + q_{\mathsf{A}_1}(x)$ *for any* $x \in F$ *such that* $u_1(x) \neq 0$, $p_1(x) = p_2(x)$ *for any* $x \in H_1$ *such that* $u_1(x) \neq 0$ *and moreover* $c^1_F(x,y) = c^2_F(x,y)$ *for any* $x, y \in \bar{F}$ *such that* $u_1(x) \neq u_1(y)$.

Proof. If $g \in \mathcal{C}(H_2)$ is given by $g(x) = u_1(x) = u_2(x)$, $x \in H_2$, then u_1 and u_2 are respectively the unique solutions of the mixed Dirichlet-Robin boundary value problems

$$-\mathsf{div}\,(\mathsf{A}_i\mathsf{d}u)(u) + q_i u = 0, \quad \text{on } F, \qquad \frac{\partial u_i}{\partial \mathsf{n}_{\mathsf{A}_i}} + p_i u_i = 0, \quad \text{on } H_1 \quad \text{ and } \quad u_i = g, \quad \text{on } H_2 \quad i = 1, 2.$$

Therefore, if we consider the quadratic forms $\mathcal{Q}_1, \mathcal{Q}_2 \colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ defined as

$$\mathcal{Q}_i(u) = \frac{1}{2}\int_{\bar{F}\times\bar{F}} c^i_F(x,y)\Big(u(x) - u(y)\Big)^2 dxdy + \int_F (q_i + q_{\mathsf{A}_i})\, u^2 d\nu + \int_{H_1} p_i u^2 d\nu, \quad i = 1, 2$$

then by applying the Dirichlet Principle, we know that u_i minimizes $\mathcal{Q}_i$ on K_g, $i = 1, 2$. Moreover, the hypotheses imply that $\mathcal{Q}_2(u) \geq \mathcal{Q}_1(u)$ for any $u \in K_g$. In addition, identity (19) implies that

$$\mathcal{Q}_1(u_1) = \int_{H_2} u_1 \frac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}}\, d\nu = \int_{H_2} u_2 \frac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}}\, d\nu = \mathcal{Q}_2(u_2) \geq \mathcal{Q}_1(u_2)$$

and hence $u_2 = u_1$ on $\bar{F}$. Moreover, if $v = u_1 = u_2$, then

$$\begin{aligned} 0 = &\frac{1}{2}\int_{\bar{F}\times\bar{F}} \Big(c^2_F(x,y) - c^1_F(x,y)\Big)\Big(v(x) - v(y)\Big)^2 dxdy \\ &+ \int_F (q_2 + q_{\mathsf{A}_2} - q_1 - q_{\mathsf{A}_1}) v^2\, \nu\, dx + \int_F (p_2 - p_1) v^2\, \nu\, dx \end{aligned}$$

and the conclusions follow. ∎

We finish this section showing some monotonicity properties related with Problem (15) in the self-adjoint case and under hypotheses H1 and H2. We newly adapt here the corresponding results in [3] and we always suppose that it is not simultaneously verified that $H_2 = \emptyset$, $q = \dfrac{\rho\sigma}{\nu} - q_A$ on F and $p = \dfrac{\rho\sigma}{\nu}$ on H_1.

Proposition 5.9 (Hopf's minimum principle) [3, Proposition 4.6] *Let* $u \in \mathcal{C}(\bar{F})$ *such that* $\mathcal{L}(u) \geq 0$ *on* F *and* $\mathcal{U}(u) \geq 0$ *on* H_1. *Suppose that there exists* $x^* \in F$ *such that*

$$u(x^*) \leq 0 \quad \text{ and } \quad \frac{u(x^*)}{\sigma(x^*)} = \min_{x\in\bar{F}} \left\{\frac{u(x)}{\sigma(x)}\right\}.$$

Then u *coincides with a non-positive multiple of* σ, $\mathcal{L}(u) = 0$ *on* F, $\mathcal{U}(u) = 0$ *on* H_1 *and either* $u = 0$ *on* $F \cup H_1$ *or* $q = \dfrac{\rho\sigma}{\nu} - q_A$ *on* F *and* $p = \dfrac{\rho\sigma}{\nu}$ *on* H_1.

Proposition 5.10 [3, Proposition 4.10] *Let* $u \in \mathcal{C}(\bar{F})$ *such that* $\mathcal{L}(u) \geq 0$ *on* F *and* $\mathcal{U}(u) \geq 0$ *on* H_1. *If* $u \geq 0$ *on* H_2, *then* $u \geq 0$ *on* $\bar{F}$. *Moreover either* $u = 0$ *on* $F \cup H_1$ *or* $u(x) > 0$ *for any* $x \in F \cup H_1$.

6 Boundary value problems on uniform grids

In this section we apply the results of the preceding sections to the study of a boundary value problems on bi-dimensional uniform grids that shows the versatility of our vector calculus. In [2] the authors showed that on a uniform grid in the Euclidean n-space, any difference scheme with constant coefficients consistent with a second order linear differential operator of the form $-\mathsf{div}\,(K\nabla u)+\langle \mathsf{k},\nabla u\rangle + k_0 u$ and with constant coefficients, can be seen as an operator of the form $-\mathsf{div}\,(\mathsf{A}\mathsf{d}u)+\langle \mathsf{f},\mathsf{d}u\rangle+qu$, for a suitable choice of q,f,A and ν and μ. Moreover, special properties of the difference schemes such as consistency and positivity can be characterized in terms of q,f,A. We remark that although our techniques and results are in force in any dimension, for sake of simplicity we restrict us here to the bi-dimensional case.

For each $h>0$ we consider the subset in $\mathbb{R}^2$ given by $V_h=h\mathbb{Z}^2$. The vertices $x,y\in V_h$ are *adjacent* if their Euclidean distance $|x-y|$ equals h. Therefore, if $\{\mathsf{e}_1,\mathsf{e}_2\}$ denotes the standard basis of $\mathbb{R}^2$ and we define $\mathsf{e}_3=-\mathsf{e}_1$ and $\mathsf{e}_4=-\mathsf{e}_2$, then for any $x\in V_h$, the adjacent vertices to x are $x_j=x+h\mathsf{e}_j$, $j=1,\dots,4$. The set of all the edges is denoted by E_h and hence we call *bi-dimensional uniform grid of size* h, the weighted graph (Γ_h,μ,ν) where $\Gamma_h=(V_h,E_h)$ and ν,μ are the weights on V_h defined as $\mu(x)=h$ and $\nu(x)=h^2$, for any $x\in V_h$. For any $x\in V_h$ we also consider the vertices $x_{ij}=x+h\,(\mathsf{e}_i+\mathsf{e}_j)$, $1\le i\le j\le 4$, $j\ne 2+i$ and we call *stencil at* x the set $S(x)=\{y\in V_h : d(x,y)\le 2\}$, see Figure 1.

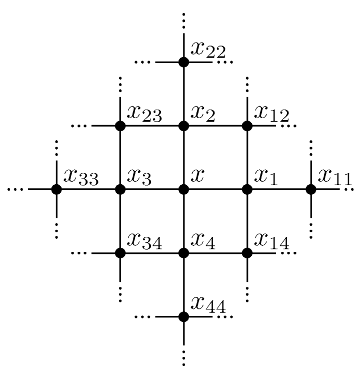

Figure 1: Bi-dimensional stencil

We say that the vector field f is the *homogeneous field determined by* $\mathsf{b}=(b_j)\in\mathbb{R}^4$ if $f(x,x_j)=\dfrac{b_j}{h}$ for all $x\in V_h$, $j=1,\dots,4$. Moreover f is a flow iff $b_{2+j}=-b_j$, $j=1,2$. In addition, we will say that a field of endomorphisms, A is *homogeneous* if there exists a square 4-matrix $A=(a_{ij})$ such that $a(x,x_i,x_j)=\dfrac{a_{ij}}{h}$, for all $x\in V_h$, $i,j=1,\dots,4$. In this case, we say that the homogeneous field A is *determined by* A. Moreover if we consider A_1,A_2,A_3,A_4 square 2-matrices such that $A=\begin{bmatrix} A_1 & A_2\\ A_3 & A_4\end{bmatrix}$ and $B=\frac{1}{2}\begin{bmatrix} A_1+A_4^* & A_2+A_3^*\\ A_3+A_2^* & A_4+A_1^*\end{bmatrix}$ in [2, Corollary 3.2] it was proved that $\mathsf{div}\,(\mathsf{A}\mathsf{d}u)=\mathsf{div}\,(\mathsf{B}\mathsf{d}u)$ which represents a discrete version of the equality between cross derivatives. So, in what follows we assume without loss of generality that $A=\begin{bmatrix} A_1 & A_2\\ A_2^* & A_1^*\end{bmatrix}$.

We call *(second order) difference scheme with constant coefficients* on $\{\Gamma_h\}_{h>0}$, a family of second order linear operators $L_h : \mathcal{C}(V_h) \longrightarrow \mathcal{C}(V_h)$ such that for any $x \in V_h$

$$L_h(u)(x) = q(h)u(x) + \sum_{j=1}^{4} \gamma_j(h)\Big(u(x) - u(x_j)\Big) + \sum_{\substack{1\le i\le j\le 4\\ j\neq 2+i}} \gamma_{ij}(h)\,\Big(u(x) - u(x_{ij})\Big), \qquad (23)$$

where $q, \gamma_j, \gamma_{ij}\colon (0,+\infty) \longrightarrow \mathbb{R}$. For the sake of simplicity, we only consider difference schemes verifying that $\gamma_{ij} = \gamma_{2+i\,2+j}$ for any $i,j = 1,2$ and $\gamma_{14} = \gamma_{23}$. We remark that, in practice, this is a non-relevant restriction.

One of the fundamental questions in Numerical Analysis is to characterize all difference schemes with constant coefficients on $\{\Gamma_h\}_{h>0}$ that are consistent with a second order differential operator with constant coefficients. Recall that given $r > 0$, the difference scheme L_h is called *r-consistent* with the differential operator L on $\{\Gamma_h\}_{h>0}$ if $L(u)(x) - \mathcal{L}_h(u)(x) = O(h^r)$, for any $x \in V_h$ and for any u smooth enough. In the literature the study of consistency is usually performed from the expression (23), but this process is quite intricate, in general. We take advantage by expressing the difference scheme as a difference operator of the form $-\mathsf{div}\,(\mathsf{A}\; du) + \langle \mathsf{f}, \mathsf{d}u\rangle + q\,u$ and characterizing consistency in terms of q, f and A.

Proposition 6.1 ([2, Proposition 4.8]) *If L_h is a difference scheme with constant coefficients, then there exist a unique function q, a unique homogeneous field of endomorphisms A and a unique homogeneous flow f such that*

$$L_h(u) = -\mathsf{div}\,(\mathsf{A}\mathsf{d}u) + \langle \mathsf{f}, \mathsf{d}u\rangle + q\,u.$$

Moreover, if $K = (k_{ij}) \neq 0$ is a symmetric matrix, then all difference schemes that are 2-consistent with the second order differential operator with constant coefficients

$$L(u) = -k_{11}u_{xx} - 2k_{12}u_{xy} - k_{22}u_{yy} + k_1 u_x + k_2 u_y + k_0 u$$

have the expression

$$\mathcal{L}_h(u) = -\mathsf{div}\,(\mathsf{A}\mathsf{d}u) + \langle \mathsf{f}, \mathsf{d}u\rangle + k_0 u,$$

where f is the flow determined by $\dfrac{k_j}{2}$, $j = 1,2$ and A is the homogeneous field of endomorphisms determined by $A = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}$, where M is an arbitrary symmetric matrix. In addition, it is possible to choose M in such a way that $-\mathsf{div}\,(\mathsf{A}\mathsf{d}u)$ is the Laplace-Beltrami operator corresponding to a metric on Γ_h iff L is a semi-elliptic operator, whereas it is possible to choose M in such a way that the pair (A, μ) is strongly elliptic on V_h iff $k_0 \geq 0$ and $\min\{k_{11}, k_{12}\} > |k_{12}|$.

Our next aim is to analyze the consistence of the discrete boundary value problem that approximate the following Robin boundary value problem on the unit square $S = [0,1]\times[0,1]$

$$L(u) = -k_{11}u_{xx} - 2k_{12}u_{xy} - k_{22}u_{yy} + k_1u_x + k_2u_y + k_0u = g \text{ on } (0,1)\times(0,1)$$

$$U(u) = -k_{12}u_x - (k_{22}+k_2)u_y + k_0u = g_1 \text{ on } [0,1]\times\{0\}$$

$$U(u) = k_{12}u_x + (k_{22}+k_2)u_y + k_0u = g_1 \text{ on } [0,1]\times\{1\}$$

$$U(u) = -(k_{11}+k_1)u_x - k_{12}u_y + k_0u = g_1 \text{ on } \{0\}\times[0,1]$$

$$U(u) = (k_{11}+k_1)u_x + k_{12}u_y + k_0u = g_1 \text{ on } \{1\}\times[0,1]$$

For this, we consider $n \in \mathbb{N}^*$, $h = \dfrac{1}{n}$ and the set

$$F_h = \{h(i,j) \in h\mathbb{Z}^2 : i,j = 1,\dots,n-1\} \subset V_h.$$

In addition, we redefine the weight ν on the vertex boundary of F_h, in such a way that $\nu(x) = h$ for any $x \in \delta(F_h)$. This is due to the fact that the boundary has lower dimension than the interior of the set, and the value h^2 makes reference to the area of a cell.

Then, the following identities hold:

$$\overset{\circ}{F}_h = \{h(i,j) \in h\mathbb{Z}^2 : i,j = 2,\dots,n-2\},$$
$$\delta(F_h) = \{h(0,j), h(n,j), h(j,0), h(j,n) : j = 1\dots,n-1\}.$$

Moreover, the vertex boundary can be partitioned into two disjoint sets. The *corner set* of $\delta(F_h)$ is $C(F_h) = \{h(0,j), h(n,j), h(j,0), h(j,n) : j = 1, n-1\}$ and the set of *typical nodes* of the boundary, is $\delta(F_h) \setminus C(F_h)$, see Figure 2.

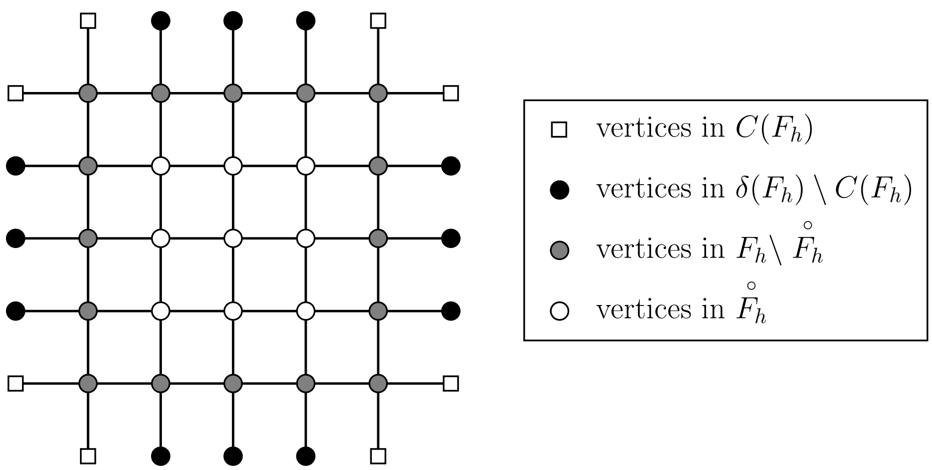

Figure 2: Different types of vertices

Given $u \in \mathcal{C}(\bar{F}_h)$, the scheme $\mathcal{L}_h(u)$ given in Proposition 6.1 has the following expression for any $x \in F_h$:

$$\begin{aligned}\mathcal{L}_h(u)(x) &= \frac{1}{h^2}\sum_{j=1}^{2}\Big(\sum_{i=1}^{2}(k_{ij}+2m_{ij})\Big)\Big(2u(x)-u(x_j)-u(x_{2+j})\Big)\\ &\quad - \frac{m_{12}}{h^2}\Big(2u(x)-u(x_{12})-u(x_{34})\Big) - \frac{(k_{12}+m_{12})}{h^2}\Big(2u(x)-u(x_{14})-u(x_{23})\Big)\\ &\quad - \sum_{j=1}^{2}\frac{m_{jj}}{2h^2}\Big(2u(x)-u(x_{jj})-u(x_{2+j2+j})\Big) + \sum_{j=1}^{2}\frac{k_j}{2h}\Big(u(x_j)-u(x_{2+j})\Big) + k_0u(x).\end{aligned}$$

On the other hand, the expression of the corresponding discrete boundary operator,

$$\mathcal{U}_h(u) = \frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}} + \langle \mathsf{f}_F, \mathsf{d}u\rangle + k_0 u,$$

is given by:

If $x = h(i,0)$, $i = 2,\dots,n-2$, then

$$\mathcal{U}_h(u)(x) = \frac{1}{h}\Big(k_{22} + k_{12} + 2(m_{22} + m_{12}) - \frac{k_2}{2}\Big)\Big(u(x) - u(x_2)\Big) - \frac{m_{22}}{2h}\Big(u(x) - u(x_{22})\Big)$$
$$- \frac{m_{12}}{h}\Big(u(x) - u(x_{12})\Big) - \frac{(m_{12} + k_{12})}{h}\Big(u(x) - u(x_{23})\Big) + k_0 u(x).$$

If $x = h(1,0)$, then

$$\mathcal{U}_h(u)(x) = \frac{1}{h}\Big(k_{22} + k_{12} + 2(m_{22} + m_{12}) - \frac{k_2}{2}\Big)\Big(u(x) - u(x_2)\Big) - \frac{m_{22}}{2h}\Big(u(x) - u(x_{22})\Big)$$
$$- \frac{m_{12}}{h}\Big(u(x) - u(x_{12})\Big) + k_0 u(x).$$

If $x = h(n-1,0)$, then

$$\mathcal{U}_h(u)(x) = \frac{1}{h}\Big(k_{22} + k_{12} + 2(m_{22} + m_{12}) - \frac{k_2}{2}\Big)\Big(u(x) - u(x_2)\Big) - \frac{m_{22}}{2h}\Big(u(x) - u(x_{22})\Big)$$
$$- \frac{(m_{12} + k_{12})}{h}\Big(u(x) - u(x_{23})\Big) + k_0 u(x).$$

The value of the boundary operator in the rest of boundary vertices can be obtained analogously. The pair $(\mathcal{L}_h,\mathcal{U}_h)$ is called *difference scheme* on $\{\bar{F}_h\}_{h>0}$.

Proposition 6.2 *Under the above conditions, for any u smooth enough, the difference scheme $(\mathcal{L}_h,\mathcal{U}_h)$ on $\{\bar{F}_h\}_{h>0}$ verifies the following properties:*

(i) $L(u)(x) - \mathcal{L}_h(u)(x) = O(h^2)$ for any $x \in F_h$.

(ii) $U(u)(x) - \mathcal{U}_h(u)(x) = O(1)$ for any $x \in \delta(F_h)$.

(iii) $U(u)(x) - \mathcal{U}_h(u)(x) = O(h)$ for any $x \in \delta(F_h) \setminus C(F_h)$ iff $k_1 = k_2 = 0$ and $M = m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(iv) $U(u)(x) - \mathcal{U}_h(u)(x) = O(h)$ for any $x \in \delta(F_h)$ iff $k_1 = k_2 = 0$, $k_{12} = 0$ and $M = 0$.

Proof. Note that (i) is a direct consequence of Proposition 6.1. The rest of results are based on the Taylor expansion of u at any node of the stencil,

$$u(x + h_1, y + h_2) = u(x,y) + h_1 u_x(x,y) + h_2 u_y(x,y)$$
$$+ \frac{h_1^2}{2} u_{xx}(x,y) + h_1 h_2 u_{xy}(x,y) + \frac{h_2^2}{2} u_{yy}(x,y) + O(h_1^3 + h_2^3).$$

So, (ii) is obvious. If $z = h(i,0)$, $i = 2,\dots,n-2$, then

$$U(u)(z) - \mathcal{U}_h(u)(z) = \frac{1}{2}\,(2m_{22} - 3k_2)\,u_y(z) - \frac{h}{2}\,(2m_{12} + k_{12})\,u_{xx}(z)$$
$$- \frac{h}{4}\,(2k_{22} + k_2)\,u_{yy}(z) + hk_{12}u_{xy}(z) + O(h^2).$$

Therefore, $U(u)(z)-\mathcal{U}_h(u)(z)=O(h)$ iff $m_{22}=\frac{3k_2}{2}$. If $z=h(i,n)$, $i=2,\dots,n-2$, then

$$U(u)(z)-\mathcal{U}_h(u)(z)=\frac{1}{2}\left(k_2-2m_{22}\right)u_y(z)-\frac{h}{2}\left(2m_{12}+k_{12}\right)u_{xx}(z)$$
$$+\frac{h}{4}\left(2k_{22}+k_2\right)u_{yy}(z)+hk_{12}u_{xy}(z)+O(h^2).$$

Therefore, $U(u)(z)-\mathcal{U}_h(u)(z)=O(h)$ iff $m_{22}=\frac{k_2}{2}$. In conclusion, $m_{22}=k_2=0$. Reasoning analogously for nodes $z=h(0,j)$ and $z=h(n,j)$, $j=2,\dots,n-2$, we conclude (iii).

Under conditions of (iii), if we consider $z=h(1,0)$, then

$$U(z)-\mathcal{U}_h(u)(z)=(m+k_{12})\Big(u_x(z)-u_y(z)\Big)-\frac{h}{2}\,m\,u_{xx}(z)$$
$$+\frac{h}{2}\left(m+k_{12}+k_{22}\right)u_{yy}(z)-h\,m\,u_{xy}(z)+O(h^2).$$

Therefore, $U(u)(z)-\mathcal{U}_h(u)(z)=O(h)$ iff $m=-k_{12}$. Moreover, if $z=h(n-1,0)$, then

$$U(z)-\mathcal{U}_h(u)(z)=m\Big(u_x(z)+u_y(z)\Big)-\frac{h}{2}\left(m+k_{12}\right)u_{xx}(z)$$
$$+\frac{h}{2}\left(m+k_{22}\right)u_{yy}(z)+h(m+k_{12})u_{xy}(z)+O(h^2)$$

and hence, $U(u)(z)-\mathcal{U}_h(u)(z)=O(h)$ iff $m=0$. Therefore, $k_{12}=0$ and the claim (iv) follows since the rest of vertices of $C(F_h)$ do not introduce more conditions. ∎

To end the paper we assume the hypotheses of Proposition 6.2(iii) and we show the value of m for the most commonly used difference schemes, see [2] and the references therein. Recall that in this case, the fact that the pair (A,μ) is strongly elliptic is equivalent to the fact that the scheme $(\mathcal{L}_h,\mathcal{U}_h)$ is of positive type. In any case, we only show the expression for $\mathcal{U}_h$ in nodes of the form $x=h(i,0)$, $i=1,\dots,n-1$, since the expression in the rest of nodes is analogue.

The value $m=0$ corresponds to the standard difference scheme, that is of positive type iff $k_0\ge 0$ and K is a strictly diagonally dominant M-matrix, s.d.d M-matrix in short. So, if $x\in F_h$,

$$\mathcal{L}_h(u)(x)=\frac{1}{h^2}\sum_{j=1}^{2}\Big(\sum_{i=1}^{2}k_{ij}\Big)\Big(2u(x)-u(x_j)-u(x_{2+j})\Big)-\frac{k_{12}}{h^2}\Big(2u(x)-u(x_{14})-u(x_{23})\Big)+k_0u(x).$$

If $x=h(i,0)$, $i=2,\dots,n-1$, then

$$\mathcal{U}_h(u)(x)=\frac{1}{h}\Big(k_{22}+k_{12}\Big)\Big(u(x)-u(x_2)\Big)-\frac{k_{12}}{h}\Big(u(x)-u(x_{23})\Big)+k_0u(x),$$

whereas if $x=h(1,0)$, then $\mathcal{U}_h(u)(x)=\dfrac{1}{h}\Big(k_{22}+k_{12}\Big)\Big(u(x)-u(x_2)\Big)+k_0u(x)$. In particular, when K is diagonal the scheme is the well-known 4 point scheme.

The value $m=-k_{12}$ corresponds to the difference scheme

$$\mathcal{L}_h(u)(x)=\frac{1}{h^2}\sum_{j=1}^{2}\Big(k_{jj}-k_{12}\Big)\Big(2u(x)-u(x_j)-u(x_{2+j})\Big)+\frac{k_{12}}{h^2}\Big(2u(x)-u(x_{12})-u(x_{34})\Big)+k_0u(x).$$

If $x = h(i,0)$, $i = 1,\dots,n-2$, then

$$\mathcal{U}_h(u)(x) = \frac{1}{h}\Big(k_{22} - k_{12}\Big)\Big(u(x) - u(x_2)\Big) + \frac{k_{12}}{h}\Big(u(x) - u(x_{12})\Big) + k_0 u(x),$$

whereas if $x = h(n-1,0)$, then $\mathcal{U}_h(u)(x) = \frac{1}{h}\Big(k_{22}-k_{12}\Big)\Big(u(x)-u(x_2)\Big)+k_0u(x)$. The above scheme is of positive type iff $k_0 \ge 0$ and K is a non-negative and s.d.d. matrix.

When, K is a diagonal matrix, the value $m = -\frac{(k_{11}+k_{22})}{12}$ leads to the nine-point difference scheme

$$\mathcal{L}_h(u)(x) = \frac{(5k_{11} - k_{22})}{6h^2}\Big(2u(x) - u(x_1) - u(x_3)\Big) + \frac{(5k_{22} - k_{11})}{6h^2}\Big(2u(x) - u(x_2) - u(x_4)\Big)$$
$$+ \frac{(k_{11} + k_{22})}{12h^2}\Big(4u(x) - u(x_{12}) - u(x_{34}) - u(x_{23}) - u(x_{14})\Big) + k_0 u(x).$$

If $x = h(i,0)$, $i = 2,\dots,n-2$, then

$$\mathcal{U}_h(u)(x) = \frac{(5k_{22} - k_{11})}{6h}\Big(u(x) - u(x_2)\Big) + \frac{(k_{11} + k_{22})}{12h}\Big(2u(x) - u(x_{12}) - u(x_{23})\Big) + k_0 u(x).$$

If $x = h(1,0)$, then

$$\mathcal{U}_h(u)(x) = \frac{(5k_{22} - k_{11})}{6h}\Big(u(x) - u(x_2)\Big) + \frac{(k_{11} + k_{22})}{12h}\Big(u(x) - u(x_{12})\Big) + k_0 u(x).$$

If $x = h(n-1,0)$, then

$$\mathcal{U}_h(u)(x) = \frac{(5k_{22} - k_{11})}{6h}\Big(u(x) - u(x_2)\Big) + \frac{(k_{11} + k_{22})}{12h}\Big(u(x) - u(x_{23})\Big) + k_0 u(x).$$

If we suppose that $k_{11} \le k_{22}$, then the above scheme is of positive type iff $k_0 \ge 0$, $k_{11} > 0$ and $5k_{11} > k_{22}$.

Finally, if $K = kI$, where $k \ne 0$ and I is the 2-order identity matrix, then the following difference scheme is the so-called cross scheme and correspond to the choice $m = -\frac{k}{2}$.

$$\mathcal{L}_h(u)(x) = \frac{k}{2h^2}\Big(4u(x) - u(x_{12}) - u(x_{34}) - u(x_{23}) - u(x_{14})\Big) + k_0 u(x).$$

If $x = h(i,0)$, $i = 2,\dots,n-2$, then

$$\mathcal{U}_h(u)(x) = \frac{k}{2h}\Big(2u(x) - u(x_{12}) - u(x_{23})\Big) + k_0 u(x).$$

If $x = h(1,0)$, then $\mathcal{U}_h(u)(x) = \frac{k}{2h}\Big(u(x) - u(x_{12})\Big) + k_0 u(x)$, whereas if $x = h(n-1,0)$, then $\mathcal{U}_h(u)(x) = \frac{k}{2h}\Big(u(x) - u(x_{23})\Big) + k_0 u(x)$. Observe that the above three point formula for $\mathcal{U}_h$ on $\delta(F_h) \setminus C(F_h)$ was already obtained in [6].

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